MATH3354 Revision

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A complex Lie algebra $\mathfrak{g}$ is **reductive** if it is the complexification of the lie algebra of a compact matrix Lie group. It is **semisimple** if it is reductive and has trivial centre. If $\mathfrak{g}$ is reductive, there is an inner product on $\mathfrak{g}$ such that

$$\langle [X, Y], Z\rangle = \langle Y, [X^*, Z]\rangle.$$

A Lie algebra is **simple** if it is semisimple and has no nontrivial ideals. Every semisimple Lie algebra is the direct product of simple Lie algebras.

A **Cartan subalgebra** $\mathfrak{h} \subset \mathfrak{g}$ is a maximal commutative subalgebra where ad_H is diagonalizable. In a semisimple Lie algebra, every maximal commutative subalgebra is a Cartan subalgebra. The **rank** of a Li algebra is the dimension of any Cartan subalgebra.

An element $\alpha \in \mathfrak{h}$ is a **root** if $\exists X \in \mathfrak{g}$ so that

$$[H, X] = \langle \alpha, H\rangle X \quad \forall H \in \mathfrak{h}.$$

The set of such **root vectors** X is called $\mathfrak{g}_\alpha$ and is called the **root space.** Each $\mathfrak{g}_\alpha$ is one-dimensional, and the roots spaces are orthogonal to each other and as an inner product space

$$\mathfrak{g} = \mathfrak{h} \oplus \text{root spaces}.$$

If α is a root, $\lambda\alpha$ is a root iff $\lambda = \pm 1$.

$$[g_\alpha, g_\beta] \subset g_{\alpha+\beta}.$$

If $X \in g_\alpha$, then $X^* \in g_{-\alpha}$. For every root α, there is a $X \in \mathfrak{g}_\alpha$ such that

$$[H_\alpha, X_\alpha] = 2X_\alpha, \quad [X_\alpha, X_\alpha^*] = H_\alpha.$$

Then we find that

$$H_\alpha = \frac{2}{\langle \alpha, \alpha\rangle}\alpha$$

and this is called a **coroot**.

The **Weyl group** W of the roots R is defined as the group generated by the mappings

$$s_\alpha H = H - \frac{\alpha, H}{\alpha, \alpha}\alpha.$$

This group preserves the roots.

A **root system** is a finite set of nonzero elements of a real product space E with

1. R spans E
2. If $\alpha \in R$ and $\lambda\alpha \in R$, the $\lambda = \pm 1$
3. If $\alpha, \beta \in R$ then $s_\alpha \beta \in R$
4. For all $\alpha, \beta \in R$,
$$\frac{2\langle \alpha, \beta\rangle}{\langle \alpha, \alpha\rangle} \in \mathbb{Z}.$$

There is a one-to-one correspondence between semisimple lie algebras and root systems. The **rank** of a the system is the dimension of E.

Suppose α, β are root which are not multiples of each other, and assume $\langle \beta, \beta\rangle \leq \langle \alpha, \beta\rangle$. Then one of the following is true

1. $\langle \alpha, \beta\rangle = 0$
2. $\|\alpha\|^2 = \|\beta\|^2$ and the angle between α and β is $\pi/3$ or $2\pi/3$
3. $\|\alpha\|^2 = 2\|\beta\|^2$ and the angle between α and β is $\pi/4$ or $3\pi/4$
4. $\|\alpha\|^2 = 3\|\beta\|^2$ and the angle between α and β is $\pi/6$ or $5\pi/6$

A **base** for R is a set of roots which form a basis for E, and for which every root can be written as the sum of all positive or all negative elements of the base.

An integral element $\mu \in E$ is a vector such that for all $\alpha \in R$,
$$\langle \mu, H_\alpha\rangle = 2\frac{\langle \mu, \alpha\rangle}{\langle \alpha, \alpha\rangle} \in \mathbb{Z}.$$

For a give base α_i, fundamental weights are the weights such that
$$2\frac{\langle \mu_i, \alpha_j\rangle}{\langle \alpha_j, \alpha_j\rangle} = \delta_{ij}$$

since any other weight can be built from sums of these weights. An element $\mu \in E$ is **dominant** if $\langle \alpha, \mu\rangle \geq 0$ and strictly dominant if $\langle \alpha, \mu\rangle > 0$. An element μ is **higher** than ν if $\mu - \nu$ can be written as positive sum of elements in the base.

If (π, V) is a representation of $\mathfrak{g}$, then a **weight vector** is a $v \in V$ such that
$$\pi(H)v = \langle \lambda, H\rangle v$$

for some $\lambda \in \pi(\mathfrak{h})$. We then call λ a **weight**. Every weight in a finite representation must be an integral element of $\mathfrak{h}$. Every irreducible finite-dimensional representation of $\mathfrak{g}$ has a unique highest weight. Conversly, every dominant integral element is a highest weight of an irreducible representation.