Revision Notes for Algebraic Topology

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1 General Notions

1.1 Operations on Spaces

Given two spaces X, Y, the simplest operations are the product of spaces $X \times Y$, and disjoin union $X \amalg Y$. If we have $A \subset X$, the quotient X/A can be constructed, with quotient map $q : X \to X/A$.

The cone CX is $X \times I/X \times \{0\}$, and the suspension SX is $X \times I/X \times \{0, 1\}$. Maps between X and Y can naturally be extended to SX and SY likewise CX and CY.

For pointed spaces (X, x_0) and (Y, y_0), the wedge sum $X \vee Y$ is the space $X \amalg Y/(x_0 \sim y_0)$. We can embed this in $X \times Y$ via the map $f(x) = (x, y_0)$, $f(y) = (y, x_0)$. The smash product is then defined as $X \times Y/X \vee Y$.

1.2 Homotopy

A **homotopy** is a map $F : X \times I \to Y$. Two maps f_0, f_1 are **homotopic** if a homotopy connects them, in which case we write $f_0 \simeq f_1$. If between two spaces X and Y there are maps $f : X \to Y$ and $g : Y \to X$ so that $fg \simeq 1$, $gf \simeq 1$, then we say that X and Y are **homotopically equivalent**. A map is **nullhomotopic** if it is equivalent to a constant map.

A **retraction** from X to a subspace $A \in X$ is a map with $r(X) = A$, $r|_A = \mathrm{id}_A$. A **deformation retraction** is a homotopy from id_X to a retraction. If the identity on X is nullhomotopic, then we say that X is **contractible**. If f_t is a homotopy whose restriction to A is independent on t, then we say that f_t is homotopy relative to A, or rel A for short.

Let $f_0 : X \to Y$ and for $A \subset X$, let $f_t : A \to Y$ be a homotopy of $f_0|_A$. If for this setup there is always an $f_t : X \to Y$ then we say the pair (X, A) have the **homotopy extension property.**

Proposition: If (X, A) satisfy the homotopy extension property, then the quotient map $q : X \to X/A$ is a homotopy equivalence.

Proposition: Suppose (X, A) and (Y, A) satisfy the homotopy extension property, and $f : X \to Y$ is a homotopy equivalence with $f|_A = \mathrm{id}_A$. The f is a homotopy equivalence rel A.

1.3 CW Complexes

Cell complexes describe ways to create spaces by gluing together discs. They are made through the following steps:

1. Start with a discrete set X^0.
2. For the n-skeleton X^n, construct X^{n+1} by attaching **n-cells** e^n_α which are open disks of dimension n. We have $\Phi : \amalg\, \partial e^n_\alpha \to X^n$ which identifies the boundaries of the n-cells with the X^{n-1} skeleton, and then construct

$$X^{n+1} = X^n \amalg e^n_\alpha / x \sim \Phi(x).$$

If $X = X^n$ for some n, then X is said to have dimension n. A **subcomplex** of X is a closed subspace A which is a union of cells of X. For a complex and subcomplex, (X, A) has the homotopy equivalence property.

1.4 Important Spaces

The most basic space is the disc $D^n = \{x \in \mathbb{R}^n | x| \leq 1\}$. The n-sphere can be defined as

$$S^n = \partial D^{n+1} = D^n / \partial D^n.$$

Real projective space is defined as

$$\mathbb{R}P^n = \mathbb{R}^{n+1}/(x \sim \lambda x) = S^n/(x \sim -x) = D^n/\{x = -x \text{ iff } |x| = 1\}.$$

Since we can construct $\mathbb{R}P^n$ from $\mathbb{R}P^{n-1}$ by attaching an n-cell e_n via the quotient map $S^{n-1} \to \mathbb{R}P^{n-1}$. Hence itt has cell structure $e_0 \cup e_1 ... \cup e_n$. Likewise for complex projective space

$$\mathbb{C}P^n = \mathbb{C}^n/(x \sim \lambda x)$$

which has cell complex stucture $e^0 \cup e^2 \cup ... \cup e^{2n}$ attached via the quotient map $S^{2n-1} \to \mathbb{C}P^{n-1}$.

2 Fundamental Groups

2.1 Basics

Consider the set of paths $I \to X$ denoted X^I. We shall consider homotopy classes of such paths which preserve the endpoints. Firstly, any reparameterization of $f \in X^I$ remains in the same homotopy class $[f]$. So if $f(1) = g(0)$, then we can compose the two paths

$$f \cdot g(t) = f(2t) \text{ for } t \leq \frac{1}{2}, \quad f \cdot g(t) = g(2t + 0.5) \text{ for } t > \frac{1}{2}$$

and then define $[f] \cdot [g] = [f \cdot g]$. The composition is associative, and for any f, there is an inverse $\bar{f}(t) = f(1 - t)$, since $[\bar{f} \cdot f] = [f(0)]$. Hence the set of homotopy classes is a groupoid.

A loop is a path for which $f(0) = f(1)$. The **fundamental group** $\pi_1(X, x_0)$ is the set of homotopy classes of loops starting at x_0, with composition as defined above. If X is path-connected, then any two $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic, since we can take a path h from x_0 to x_1 and then $[h]\pi_1(X, x_0)[\bar{h}] = \pi_1(X, x_1)$. We say X is **simply-connected** if it is path-connected with trivial fundamental group.

The map π_1 from the category of pointed sets to the category of groups is a functor, with continuous maps $f : X \to Y$ inducing homomorphism $f_* : \pi_1(X) \to \pi_1(Y)$ via $f_*[\phi] = [f(\phi)]$. This takes commutative diagrams of topological spaces into commutative diagrams of groups. A simple corollary of this is that injective functions induce injective homomorphism, and homeomorphisms induce isomorphisms.

Proposition: For path-connected X, Y, $\pi_1(X \times Y) \approx \pi_1(X) \times \pi_1(Y)$.

Proposition: For path-connected X, Y, $\pi_1(X \vee_{x_0} Y) \approx \pi_1(X) * \pi_1(Y)$ if x_0 is a deformation retraction of some open neighbourhood of x_0 in both X and Y.

The functor π_1 is homotopy invariant; ie, if there is a homotopy equivalence between X and Y, then $\pi_1(X) \approx \pi_1(Y)$. Furthermore, if ϕ_t is a homotopy, then $\phi_{0*} = \phi_{1*}$. So we can consider π_1 to be a functor on the homotopy category.

2.2 Some Basic Fundamental Groups

Proposition: The fundamental group of $\mathbb{R}^n$ is trivial. To see this, note that for any two loops $f(s)$ and $g(s)$, the linear combination

$$F_t(s) = tf(s) + (1 - t)g(s)$$

is a homotopy between the loops.

Theorem: The fundamental group of S^1 is $\mathbb{Z}$.

This can be used to prove three interesting results by contradiction, through the construction of a homotopy between a trivial and a non-trivial loop in S^1.

Fundamental Theorem of Algebra: Every non-constant polynomial has a root.

Brouwer Fixed Point Theorem: Every map from $D^2 \to D^2$ has a fixed point.

Two-Dimensional Borsuk-Ulam Theorem: For every continuous mapping $f : S^2 \to \mathbb{R}^2$, there exists a pair of antipodal points x and $-x$ so that $f(x) = f(-x)$.

As a corollary, if S^2 is expressed as the union of three closed sets, then at least one must contain a pair of antipodal points.

Proposition: For $n \geq 2$, the sphere S^n is simply-connected.

2.3 Van Kampen Theorem

Let X be the union of path-connected open sets A_α, each containing $x_0 \in X$. For each A_α we have an inclusion $j_\alpha : A_\alpha \hookrightarrow X$ inducing a homomorphism $j_{\alpha*}$ between the fundamental groups. This can be extended to a homomorphism $\Phi : *_\alpha \pi_1(A_\alpha, x_0) \to \pi_1(X, x_0)$. The first part of the Van

Kampen Theorem states that if each intersection $A_i \cap A_j$ is path-connected, then Φ is surjective.

Let $i_{\alpha\beta_*} : \pi_1(A_\alpha \cap A_\beta) \to \pi_1(A_\alpha)$. Since $j_\alpha i_{\alpha\beta} = j_\beta i_{\beta\alpha}$, the kernel of Φ must include elements of the form $i_{\beta\alpha_*}^{-1}(\omega) i_{\alpha\beta_*}(\omega)$. The second part of the Van Kampen theorem states that if $A_\alpha \cap A_\beta \cap A_\gamma$ is path connected, then the kernel N of Φ is generated by elements of form $i_{\beta\alpha_*}^{-1}(\omega) i_{\alpha\beta_*}(\omega)$, and so $\pi_1(X) \approx *_\alpha \pi_1(A_\alpha)/N$.

The wedge product $\pi_1(X \vee Y) \approx \pi_1(X) * \pi_1(Y)$ is the special case with only two sets, disjoint except for x_0.

The Van Kampen Theorem is very useful for analysing the fundamental group of 2-dimensional cell complexes. Suppose we produce a space Y by attaching 2-cells e^2_α to X via mappings $\Phi_\alpha : \nabla e^2_\alpha \to X$.

Proposition: The inclusion $X \hookrightarrow Y$ induces a surjection $\pi_1(X) \to \pi_1(Y)$ whose kernel is $\pi_1(\Phi(\amalg e^2_\alpha))$.

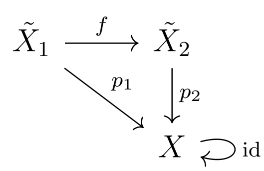
Corollary: The fundamental group of the genus g torus is $\pi_1(M_g) = \langle a_1, b_1, ...a_g, b_g | [a_1, b_1]...[a_g, b_g] \rangle$.

Corollary: For every group G there is a 2-dimensional cell complex X_G with $\pi_1(X_G) \approx G$.

2.4 Covering Spaces

Definition: A covering space of X is a space $\tilde{X}$ and a map $p : \tilde{X} \to X$ so that there is an open cover $\{U_\alpha\}$ of X with $p^{-1}(U_\alpha)$ being a disjoint union of open sets, each of which is mapped homeomorphically to U_α by p.

We can define a category where the objects are covering spaces of X along with their maps $p : \tilde{X} \to X$, and the morphisms are continuous maps $f : \tilde{X}_1 \to \tilde{X}_2$ so that $p_1 = f \circ p_2$. Since X covers itself with the identity map, every p is a morphism in this category.



This gives us a natural notion of isomorphism and automorphism; the automorphisms of a covering space are known as the **deck transformations**, $\text{aut}(\tilde{X})$.

The cardinality of $p^{-1}(x)$ is locally constant, so if X is connected, this is constant over all of X and is known as the number of sheets of X.

Given $f : Y \to X$, a **lift** of f is a map $\tilde{f} : Y \to \tilde{X}$ so that $p \circ \tilde{f} = f$.

Proposition If Y is connected, then lifts of f are unique in the sense that if two lifts agree at a single point, then they must agree everywhere.

Proposition If we have a homotopy $f_t : Y \to X$ and a lift $\tilde{f}_0$ of f_0, then there is a unique lift $\tilde{f}_t$ of the entire homotopy.

This second proposition allows us to prove

Proposition: The map covering space map $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is injective.

This proposition shows us that π_1 is a functor from the category of covering spaces of X to the category of subgroups of $\pi_1(X)$. The second category is the category where objects are subgroups of $\pi_1(X)$ along with their inclusion homomorphisms, and morphisms are maps that commute with these inclusion homomorphisms. Note this definition is identical to that of the category of covering spaces, but with X changed to $\pi_1(X)$ and covering space maps replaced with inclusion homomorphisms. Note also that the notion of isomorphism in the category of subgroups is stronger than that of groups; two subgroups are isomorphic only if they have the same image in the group.

Finally, we have the following **lifting criterion** which describes answers the existence question for lifts:

Proposition: Take a covering space $p : (\tilde{X}, \tilde{x}) \to (X, x)$ and a map $f : (Y, y) \to (X, x)$, where Y is path-connected and locally path-connected. Then a lift of f exists iff $f_*(\pi_1(Y, y)) \subset p_*(\pi_1(X, x))$.

If we put further restrictions on X, we can prove more about π_1. Let us assume that X is path-connected and locally path-connected, and let us examine only path-connected covering spaces. Then

Proposition: If $p_1(\tilde{x}_1) = p_2(\tilde{x}_2)$ and $p_{1*}(\pi_1(\tilde{X}_1, \tilde{x}_1)) = p_{2*}(\pi_1(\tilde{X}_2, \tilde{x}_2))$, then there is a base-point preserving isomorphism from $\tilde{X}_1$ to $\tilde{X}_2$.

Proposition: If $\tilde{x}_0, \tilde{x}_1 \in p^{-1}(x_0)$, then $p_*(\tilde{\pi}_1(\tilde{X}, \tilde{x}_0))$ and $p_*(\pi_1(\tilde{X}, \tilde{x}_1))$ are (non-trivial) conjugate subgroups.

In fact, if γ is a path between $\tilde{x}_0$ and $\tilde{x}_1$ then

$$[p(\gamma)]^{-1} p_*(\tilde{\pi}_1(\tilde{X}, \tilde{x}_1))[p(\gamma)] = p_*(\pi_1(\tilde{X}, \tilde{x}_0)).$$

We say that $p : \tilde{X} \to X$ is **normal** if for every $\tilde{x}_0, \tilde{x}_1 \in p^{-1}(x_0)$ there is a deck transformation taking $\tilde{x}_0$ to $\tilde{x}_1$. Unsurprisingly, we have

Proposition: $\tilde{X}$ is normal iff $p_*(\pi_1(\tilde{X}))$ is normal.

Proposition: $\mathrm{aut}(\tilde{X}) \approx N(\pi_1(\tilde{X}))/\pi_1(\tilde{X})$.

$N(H)$ is the normalizer of a subgroup H in G.

The above propositions show that in some sense, π_1 is a injection from the coverings of X to the subgroups of $\pi_1(X)$. To turn this into a bijection, we need to add an extra condition on X. We say X is **semilocally** simply-connected if every point in x has a neighbourhood U so that the inclusion $\pi_1(U) \to \pi_1(X)$ is trivial. This is a strictly weaker condition then local simply-connectedness.

Proposition: If X is path-connected, locally path-connected, and semilocally simply-connected, then for every $H \subset \pi_1(X)$, there is a space $p : X_H \to X$ with $p_*(\pi_1(X)) = H$. In particular, when H is the trivial subgroup, $\tilde{X}$ is known as the universal covering space.

Theorem: For path-connected, locally path-connected, and semilocally simply-connected X, there is a bijection between basepoint preserving path-connected covering spaces and subgroups of $\pi_1(X, x_0)$. Ignoring basepoints, there is a bijection between path-connected covering spaces and conjugate subgroups of $\pi_1(X)$.

Given a group G and a space Y, an **action** of G on Y is a homomorphism from G to the group of homeomorphisms of Y to itself. A **covering space action** is an action so that for every y there is a neighbourhood U with $g_1(U) \cap g_2(U)$ not empty implying $g_1 = g_2$. The **orbit space** is the space Y/G constructed by quotienting out $x \sim y$ is $x = g(y)$ for some $g \in G$.

Proposition: For a covering space action, the quotient $p : Y \to Y/G$ is a normal covering space.

Proposition: If Y is path-connected, $\text{aut}(Y/G) \approx G$.

3 Homology

3.1 Chain Complexes and Exact Sequences

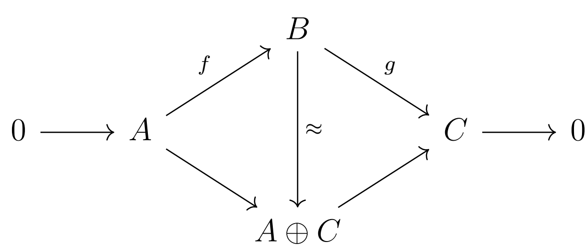
A **chain complex** is a diagram

$$... \to A_{i+1} \stackrel{\alpha_{i+1}}{\to} A_i \stackrel{\alpha_i}{\to} A_{i-1} \to ...$$

of abelian groups so that $\partial^2 = 0$. An **exact sequence** is a chain complex where $\ker \alpha_i = \text{im } \alpha_{i-1}$. A **short exact sequence** is an exact sequence

$$0 \to A \stackrel{f}{\to} B \to C \stackrel{g}{\to} 0.$$

This is equivalent to demanding that f is injective and g is surjective, so $C \approx B/f(A)$. A **split exact sequence** is an exact sequence which is part of the commutative diagram:



According to the splitting lemma, this is equivalent to the existence of a left inverse to f or a right inverse to g.

A **chain map** is a series of maps f_i so that the below diagram commutes.

$$\begin{array}{ccccccccc}
... & \xrightarrow{\alpha_{i+2}} & A_{i+1} & \xrightarrow{\alpha_{i+1}} & A_i & \xrightarrow{\alpha_i} & A_{i-1} & \xrightarrow{\alpha_{i-1}} & ... \\
 & & \downarrow{\scriptstyle f_{i+1}} & & \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{i-1}} & & \\
... & \xrightarrow{\beta_{i+2}} & B_{i+1} & \xrightarrow{\beta_{i+1}} & B_i & \xrightarrow{\beta_i} & B_{i-1} & \xrightarrow{\beta_{i-1}} & ...
\end{array}$$

3.2 Singular Homology

For a set of points $v_0, ..., v_n$ we define the **simplex** as the set of convex combinations of the points

$$[v_0, ..., v_n] = \left\{ \sum_i t_i v_i \in \mathbb{R}^n | \sum_i t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i \right\}.$$

The standard n-simplex is

$$\Delta^n = \left\{ (t_0, ..., t_n) \in \mathbb{R}^n | \sum_i t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i \right\}.$$

By deleting a v_j, we get a face of $[v_0, ..., v_n]$. The **boundary** $\partial[v_0, ..., v_n]$ is defined as the set of faces.

A **singular n-simplex** on a space X is a map $\sigma : \Delta^n \to X$. The set of n**-chains** $C_n(X)$ is defined as the free abelian group generated by the set of singular n-simplices; they are finite formal sums $\sum_i \sigma_i$ for $n_i \in \mathbb{Z}$. The **boundary map** is defined as

$$\partial_n(\sigma) = \sum_i (-1)^i \sigma_| [v_0, ..., \hat{v}_i, ..., v_n].$$

From this definition we easily can prove

Proposition: The boundary of a boundary is trivial, $\partial^2 = 0$.

Because of this, we can define the **singular homology group** $H_n(X) = \ker \partial_n / \text{im } \partial_{n+1}$.

Proposition: If X is composed of path-components X_α, then $H_n(X) \approx \oplus_\alpha H_n(X_\alpha)$.

Proposition: If X is path-connected and nonempty, then $H_0(X) \approx \mathbb{Z}$.

Proposition: If X is a point, then $H_n(X) = 0$ for $n > 0$.

We define the **reduced homology groups** $\tilde{H}_n(X)$ using the sequence

$$... \to C_1(X) \to C_0(X) \to \mathbb{Z} \to 0$$

where the map from $\varepsilon : C_0(X) \to \mathbb{Z}$ takes every 0-chain to 1. This means that $\tilde{H}_0(X) = 0$ if X is connected. Obviously, $H_0(X) \approx H_0 \oplus \mathbb{Z}$ and $H_i(X) \approx \tilde{H}_i(X)$ for $i > 0$.

Given a map $f : X \to Y$, this induces a chain map $f_\sharp$, which in turn induces a homomorphism $f_* : H_n(X) \to H_n(Y)$.

Theorem: If f and g are homotopic, then $f_* = g_*$. In particular, homotopic spaces have isomorphic homology groups.

Hence, just like the fundamental group, the homological groups are functors from the homotopy category to an algebraic category, this time the category of abelian groups.

3.3 Relative Homology

Given a space X and subspace $A \subset X$, we define the $C_n(X, A)$ to be the quotient group $C_n(X)/C_n(A)$. Since ∂ takes $C_n(A)$ to $C_{n-1}(A)$, there is an induced quotient map $\partial : C_n(X, A) \to C_{n-1}(X, A)$. We define the **relative homology group** as $H_n(X, A) = \ker \partial / \text{im } \partial$. From the definitions, we can see that elements of $H_n(X, A)$ are represented by **relative cycles** $\alpha \in C_n(X)$ with $\partial\alpha \in C_{n-1}(A)$. A relative cycle is trivial iff it is a **relative boundary**: $\alpha = \partial\beta + \gamma$ for $\beta \in C_{n+1}(X)$ and $\gamma \in C_n(A)$.

Proposition: If x is a point in X, then $H_n(X, x) \approx \tilde{H}_n(X)$.

By examining the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_n(A) & \xrightarrow{i} & C_n(X) & \xrightarrow{j} & C_n(X,A) & \longrightarrow & 0 \\
 & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \\
0 & \longrightarrow & C_{n-1}(A) & \xrightarrow{i} & C_{n-1}(X) & \xrightarrow{j} & C_{n-1}(X,A) & \longrightarrow & 0
\end{array}$$

we find (through quite an involved diagram chase) that there is an exact sequence of homology groups

$$... \to H_n(A) \to H_n(X) \to H_n(X,A) \to H_{n-1}(A) \to ...$$

Proposition: If two maps $f, g : (X,A) \to (Y,B)$ are homotopic through maps of pairs $(X,A) \to (Y,B)$ then $f_* = g_*$.

Excision Theorem: If $Z \subset A \subset X$ such that the closure of Z is contained in the interior of A, then the inclusion $(X-Z, A-Z) \hookrightarrow (X,A)$ induces an isomorphism $H_n(X-Z, A-Z) \to H_n(X,A)$ for all n.

Equivalently, for $A, B \subset X$ whose interiors cover X, the inclusion $(B, A \cap B) \hookrightarrow (X,A)$ indues isomorphism $H_n(b, A \cap B) \to H_n(X,A)$ for all n.

A **good pair** is (X,A) is a space X and a nonempty closed subspace A this is a deformation retractof some neighborhood of X. For instance, any subcomplex of a CW complex is a good pair.

Theorem: If (X,A) are a good pair, then the quotient map $q : (X,A) \to (X/A, A/A)$ induces a n isomorphism $q_* : H_n(X,A) \to H_n(X/A, A/A) \approx \tilde{H}_n(X)$.

Proposition: For a wedge sum, $\bigoplus_\alpha \tilde{H}_n(X_\alpha) \approx \tilde{H}_n(\bigvee_\alpha X_\alpha)$ if (X_α, x_α) are good pairs.

Mayer-Vietoris Sequence: If A and B are subsets in X with X equalling the union of the interiors of A and B, then there is an exact sequence

$$... \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B) \to ...$$

3.4 Simplicial Homology

Whilst singular homology can be used to define the homology of general spaces, simplicial homology provides a way of computing the homology of basic spaces known as Δ-complexes. A Δ-complex structure on a space X is a collection of maps $\sigma_\alpha : \Delta^n \to X$ so that

1. σ_α acts injectively on the interior of Δ^n, and every point in X is in the image of exactly one interior.
2. Each restriction of σ_α to a face is one of the maps $\sigma_\beta : \Delta^{n-1} \to X$.
3. A set $A \subset X$ is open iff $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each α.

We then define $\Delta_n(X)$ to be the set of mappings σ_α, and can continue defining homology from these groups just as in singular homology.

Theorem: The homology groups calculated through singular and simplicial homology are isomorphic.

By dividing spaces into Δ-complexes, we can calculate a few homologies.
Proposition: For the n-sphere, $\tilde{H}_n(S^n) = \mathbb{Z}$ and otherwise $\tilde{H}_k(S^n) = 0$.
Corollary: ∂D^n is not a retract of D^n, and hence every map $f : D^n \to D^n$ has a fixed point.
Theorem: If $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are homeomorphic, then $m = n$.
Proposition: For the torus, $\tilde{H}_1(T) = \mathbb{Z}$, $\tilde{H}_2(T) = \mathbb{Z} \oplus \mathbb{Z}$, and $\tilde{H}_i(T) = 0$ otherwise.

3.5 Map Degree and Cellular Homology

A map $f : S^n \to S^n$ induces a map $f_* : H_n(S^n) \to H_n(S^n)$ which is a homomorphism from $\mathbb{Z}$ to itself. Hence it is multiplication by an integer, this integer is known as the **degree** of the map.
Basic Properties:

1. $\deg f = 0$ if f is not surjective
2. $\deg R = -1$ for reflection R
3. If a map has no fixed points, then the degree is $(-1)^{n+1}$

Theorem: S^n has a continuous field of nonzero tangent vectors iff n is odd
Theorem: $\mathbb{Z}_2$ is the only nontrivial group that can act freely on S^n if n is even.
Theorem: $\deg f = \sum_i \deg f|_{x_i}$ where $x_i \in f^{-1}(y)$
Theorem: $\deg(Sf) = \deg f$.

For a CW complex X, the cellular chain complex is the LES

$$... \to H_{n+1}(X^{n+1}, X^n) \stackrel{d_{n+1}}{\to} H_n(X^n, X^{n-1}) \to ...$$

The homology of this sequence is the same as the homology of X. We can calculate

$$d_n(e^n_\alpha) = \sum_\beta d_{\alpha\beta} e^{n-1}_\beta$$

where $d_{\alpha\beta}$ is the degree of the map $S^{n-1}_\alpha \to X^{n-1} \to S^{n-1}_\beta$, ie the composition of the attaching map with the quotient killing everything in $X^{n-1} - e^{n-1}_\beta$. The **Euler Characteristic** is defined as

$$\chi(X) = \sum_n (-1)^n c_n = \sum_n (-1)^n \text{rank} H_n(X)$$

where c_n is the number of n-cells.
Theorem: The first three homology groups of the genus g torus are $\mathbb{Z}, \mathbb{Z}^{2g}, \mathbb{Z}$, with all others being trivial.

3.6 Formal Viewpoint

A homology theory assigns to each CW complex X a sequence of abelian groups $\tilde{h}_n(X)$ and to each $f : X \to Y$ a sequence of homomorphisms $f_* : \tilde{h}_n(X) \to \tilde{h}_n(Y)$ such that

1. If f and g are homotopic, then $f_* = g_*$.
2. There is a long exact sequence

$$... \to \tilde{h}_n(A) \to \tilde{h}_n(X) \to \tilde{h}_n(X/A) \xrightarrow{\partial} \tilde{h}_{n-1}(A) \to ...$$

   where ∂ is natural, so that for each $f : (X, A) \to (Y, B)$,

$$\begin{array}{ccc} \tilde{h}_n(X/A) & \xrightarrow{\partial} & \tilde{h}_{n-1}(A) \\ \downarrow f_* & & \downarrow f_* \\ \tilde{h}_n(Y/B) & \xrightarrow{\partial} & \tilde{h}_{n-1}(B) \end{array}$$

3. $\tilde{h}_n(\vee_\alpha X_\alpha) \approx \oplus_\alpha \tilde{h}_n(X_\alpha)$

The groups $h_n(x_0) \approx \tilde{h}_n(S^0)$ are called the coefficients of the homology theory.

4 Cohomology

4.1 Cohomology Groups

Cohomology is homology with the arrows flipped. Given the chain complex

$$... \xrightarrow{\partial} C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} ...$$

we have the dual **cochain** complex given by taking the groups $\text{Hom}(C_n, G)$; that is, the group of homomorphisms from C_n to G. This gives us

$$... \longleftarrow \text{Hom}(C_{n+1}, G) \longleftarrow \text{Hom}(C_n, G) \longleftarrow \text{Hom}(C_{n-1}, G) \longleftarrow ...$$

which can be written as

$$... \xleftarrow{\delta} C^{n+1} \xleftarrow{\delta} C^n \xleftarrow{\delta} C^{n-1} \xleftarrow{\delta} ...$$

The cohomology groups $H^n(C; G)$ are then the homology of this chain complex. The **universal coefficient theorem** shows that there is a natural split exact sequence

$$0 \to \text{Ext}(H_{n-1}(X), G) \to H^n(X; G) \to \text{Hom}(H_n(X), G) \to 0.$$

The group $\text{Ext}(H, G)$ can be calculated from the three rules

- $\text{Ext}(H \oplus H', G) \approx \text{Ext}(H, G) \oplus \text{Ext}(H', G)$
- $\text{Ext}(\mathbb{Z}, G) \approx 0$

- $\text{Ext}(\mathbb{Z}_n, G) \approx G/nG$

If two spaces have the same homology groups, they have the same cohomology groups regardless of G.

The long exact sequence of a pair, excision and the Mayer-Vietoris sequence hold with the arrows reversed. Likewise the relationship between cellular, singular, and simplicial homology holds in cohomology as well.

4.2 Cup Product

The reason cohomology is interesting is that there is natural product on cohomology groups called the **cup product.** Given a ring R and for cochains $\alpha \in C^k(X; R)$ and $\beta \in C^j(X; R)$, define

$$(\alpha \smile \beta)(\sigma) = \alpha(\sigma|[v_0, ..., v_k])\beta(\sigma|[v_k, ..., v_{k+j}])$$

where $\sigma \in C_{k+l}(X; R)$. This is associative and distributive. Furthermore,

$$\delta(\alpha \smile \beta) = (\delta\alpha) \smile \beta + (-1)^k \alpha \smile \delta\beta$$

so the cup product can be extended to the cohomology groups,

$$\smile : H^j(X) \times H^k(X) \to H^{j+k}(X).$$

We can therefore combine all of the cohomology groups into a graded cohomology ring $H^*(X; G)$, where if $\alpha \in H^j(X)$ we say that $|\alpha|$, the **dimension** of α, is j. The product is **commutative**:

$$\alpha \smile \beta = (-1)^{jk} \beta \smile \alpha.$$

Theorem: $H^*(\mathbb{R}P^n; \mathbb{Z}_2) \approx \mathbb{Z}_2[\alpha]/\alpha^{n+1}$ with $|\alpha| = 1$.
$H^*(\mathbb{C}P^n; \mathbb{Z}) \approx \mathbb{Z}[\alpha]/\alpha^{n+1}$ with $|\alpha| = 2$.
$H^*(\mathbb{R}P^\infty, \mathbb{Z}) \approx \mathbb{Z}[\alpha]/2\alpha$, with $|\alpha| = 2$.
$H^*(\mathbb{R}P^{2n}, \mathbb{Z}) \approx \mathbb{Z}[\alpha]/(2\alpha, \alpha^{k+1})$, with $|\alpha| = 2$.
$H^*(\mathbb{R}P^{2n+1}, \mathbb{Z}) \approx \mathbb{Z}[\alpha, \beta]/(2\alpha, \alpha^{k+1}, \beta^2, \alpha\beta)$, with $|\alpha| = 2, |\beta| = 2k + 1$.