How Fourier series and the method of Separation of Variables can be used to solve the one-dimensional Heat Equation

Extended Essay

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Abstract

The one-dimensional heat equation is a partial differential equation describing the flow of heat through a rod. Additionally, if we know both the initial temperature distribution along the rod combined with information on the behaviour of the rod's endpoints, then we have a boundary-value problem. This can be solved to allow us to predict the temperature of the rod over time. This essay will discuss the topic:

> *How Fourier series and the method of Separation of Variables can be used to solve the One-dimensional Heat Equation.*

We begin with a discussion of partial differentiation, followed by a derivation of the one-dimensional heat equation and explanation of its applicability to modelling other natural phenomenon. This is followed by an exploration of boundary conditions and their associated boundary-value problems. Three important boundary-value problems will be introduced, known as the Dirichlet, Neumann, and Fourier Ring problems; these will later be solved using Fourier series.

Fourier series allow us to write a function as an infinite sum of sines and cosines. The mathematics behind this is explained, and three half-range expansions of significance when solving the heat equation will be derived. Another mathematical technique known as separation of variables will then be discussed and applied to the heat equation.

We use both these mathematical techniques to derive a general series solution to each of the three boundary-value problems associated with the one-dimensional Heat Equation. The theory of Fourier series allows us to calculate the coefficients of the series using a sequence of integrals, enabling us to predict the flow of heat through the rod for each of the boundary-value problems.

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1. Introduction

The prediction of heat flow through a rod is a classical problem in mathematical physics. Heat flow is mathematically described by the one-dimensional heat equation:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \qquad k > 0$$

If we are given an initial temperature distribution along the length of the rod, with information also on the behaviour of the system's endpoints, then this forms a boundary-value problem. By solving this problem, we can predict the temperature at any point in the rod over time. This equation also describes diffusion processes so the study of it and its solutions can be applied in fields as diverse as physics, biology, and economics. For simple boundary conditions, the methods of Fourier series and Separation of Variables can be used to solve the heat equation. Our research question is thus:

> ***How Fourier series and the method of Separation of Variables can be used to solve the One-dimensional Heat Equation.***

Since the heat equation is written in terms of partial derivatives $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$, we shall begin with a description of partial differentiation, followed by a physical derivation of the one-dimensional heat equation. The application of the heat equation to other physical problems, such as diffusion, will then be explained. With this theory in place, we shall study boundary-value problems in more depth.

The methods of Fourier series and Separation of Variables are needed to solve the one dimensional heat equation, and we turn to these topics in sections 4 and 5. We then find solutions to the boundary-value problems in sections 6, 7, and 8. To calculate Fourier series, a few difficult integrals need to be evaluated. To avoid interruption, these are evaluated in Appendix A rather than in the main body of the essay.

2. The Heat Equation

2.1 Partial Derivatives

Assume we have a wire laying on the x axis between $x = 0$ and $x = L$. The temperature at each point of the wire changes over time. We define the function:

$$u(x,t), \qquad 0 \leq x \leq L$$

as the function which gives us the temperature of the point x along the wire at time t. This is a multivariable function, as it takes two inputs (x,t) and assigns them a real value $u(x,t)$. Akin to the definition of a derivative:

$$\frac{df}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

we define the *partial derivatives*:

$$\frac{\partial u}{\partial x} = u_x(x,t) = \lim_{\Delta x \to 0} \frac{u(x + \Delta x, t) - u(x,t)}{\Delta x}$$
$$\frac{\partial u}{\partial t} = u_t(x,t) = \lim_{\Delta t \to 0} \frac{u(x, t + \Delta t) - u(x,t)}{\Delta t}$$

Here $\frac{\partial u}{\partial t}$ is read as "the partial derivative of u with respect to t". Like u, both partial derivatives are functions of x and t. Physically $u_x(x,t)$ is the slope of u in the x direction at the point (x,t), and $u_t(x,t)$ is the slope of u in the t direction.

The rules of partial differentiation follow naturally from the rules of normal differentiation. If we define:

$$f(x) = u(x, t_0)$$
$$g(t) = u(x_0, t)$$

then using the definition of differentiation we find:

$$f'(x) = \left.\frac{\partial u}{\partial x}\right|_{t=t_0} = u_x(x, t_0)$$
$$g'(t) = \left.\frac{\partial u}{\partial t}\right|_{x=x_0} = u_t(x_0, t)$$

We can therefore calculate $\frac{\partial u}{\partial x}$ by finding $\frac{du}{dx}$ where t is assumed to be constant, and similarly, can calculate $\frac{\partial u}{\partial t}$ by finding $\frac{du}{dt}$ where x is held constant.

Example

Let

$$u(x,t) = x^2 t + 3xt \quad 3t + \frac{x}{t}$$

Then:

$$\frac{\partial u}{\partial x} = 2xt + 3t + \frac{1}{t} \qquad (t \text{ held constant})$$
$$\frac{\partial u}{\partial t} = x^2 + 3x - 3 - \frac{x}{t^2} \quad (x \text{ held constant})$$

A function can be partially differentiated multiple times; this is denoted by:

$$\frac{\partial^n u}{\partial x^n} \text{ or } \frac{\partial^n u}{\partial t^n}$$

An alternative notation is:

$$\frac{\partial^2 u}{\partial x^2} = u_{xx}(x,t)$$
$$\frac{\partial^2 u}{\partial t^2} = u_{tt}(x,t)$$

A Partial Differential Equation (PDE) is any equation relating an unknown function to its partial derivatives. The general theory of PDEs is not well understood; indeed, for most PDEs it is still unknown whether solutions exist. In spite of this, PDEs are ubiquitous in physics, chemistry, economics and biological modelling (James, 1993, p. 616).

2.2 Derivation of the 1-Dimensional Heat Equation

The heat equation is a simple PDE, and unlike most can be solved. We shall now show how, by using partial derivatives, the 1-dimensional heat equation can be derived from a set of physical assumptions. Assume we have a rod with a cross sectional area of A, such that:

1. The temperature throughout a cross section is constant
2. No heat escapes through the surface of the rod
3. No heat is created or destroyed within the rod
4. The rod is homogenous, with constant density ρ, thermal conductivity K and specific heat capacity c

Let the temperature at point x and time t be given by $u(x,t)$. To derive the heat equation, we shall also use the empirical laws (Zill & Cullen, 1992, pp. 771-772):

i. The heat energy per unit volume Q in an object of mass m is given by $Q = mcu$.
ii. The rate of heat flow Q_t through a cross section is proportional to the area A multiplied by u_x:

$$Q_t = -KAu_x$$

Let us now take a thin slice of the rod from x to $x + \Delta x$, and let the heat in the slice be $Q(t)$. This slice has mass:

$$m = \rho A \Delta x$$
$$\therefore Q(t) = \rho A \Delta x \, cu$$

Partially differentiating with respect to t gives:

$$Q_t = \rho \;\; \Delta xc\, u_t$$

The rate of change of heat contained in the slice is equal to the heat flowing through the ends of the slice, as no heat is created or destroyed within the slice. Thus:

$$Q_t = \left(-KAu_x(x,t)\right) - \left(-KAu_x(x+\Delta x,t)\right) = K \;\; \left(u_x(x+\Delta x,t) - u_x(x,t)\right)$$

Equating the two expressions for $Q(t)$:

$$KA\left(u_x(x+\Delta x,t) - u_x(x,t)\right) = \rho A \Delta xc\, u_t$$

$$\therefore u_t = \frac{Kc}{\rho}\frac{u_x(x+\Delta x,t) - u_x(x,t)}{\Delta x}$$

Let $k = \frac{Kc}{\rho}$, known as the *thermal diffusivity* of the material, noting that it always positive, and let $\Delta x \to 0$. From the definition of a partial derivative:

$$\therefore u_t = ku_{xx}$$

We have arrived at:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \qquad k > 0$$

and this completes our derivation.

The heat equation also describes the diffusion of a substance through a liquid. According to Fick's Second Law of diffusion:

$$\frac{\partial \phi}{\partial t} = D\frac{\partial^2 \phi}{\partial x^2}$$

where ϕ is the concentration of a dissolved substance, and D is the positive diffusion constant (Zielinski, 2006). This is another form of the heat equation. More generally, any physical process involving diffusion, whether of heat, a dissolved substance, genetic material, or information about option pricing (Rouah, 2005), can be modelled using the heat equation.

The heat equation is also important in the study of fluid dynamics, appearing in problems of one-dimensional laminar flow as well as describing the motion of compressible liquids through porous materials. Finally, the voltage $v(x,t)$ in a wire with no leakage also solves the heat equation (Carslaw & Jaeger, 1959, pp. 28-29).

2.3 Principle of Superposition

Let the functions u_1 and u_2 both solve the heat equation:

$$k\frac{\partial^2 u_1}{\partial x^2} - \frac{\partial u_1}{\partial t} = 0$$

$$k\frac{\partial^2 u_2}{\partial x^2} - \frac{\partial u_2}{\partial t} = 0$$

Adding these two equations:

$$0 = k\frac{\partial^2 u_1}{\partial x^2} - \frac{\partial u_1}{\partial t} + k\frac{\partial^2 u_2}{\partial x^2} - \frac{\partial u_2}{\partial t} = k\frac{\partial^2 (u_1 + u_2)}{\partial x^2} - \frac{\partial (u_1 + u_2)}{\partial t}$$

and therefore $u_1 + u_2$ also solve the heat equation. Now let us substitute $c_1 u_1$ into the heat equation, where c_1 is a constant:

$$k\frac{\partial^2 (c_1 u_1)}{\partial x^2} - \frac{\partial (c_1 u_1)}{\partial t} = kc_1\frac{\partial^2 u_1}{\partial x^2} - c_1\frac{\partial u_1}{\partial t} = c_1\left(k\frac{\partial^2 u_1}{\partial x^2} - \frac{\partial u_1}{\partial t}\right) = 0$$

Combining these facts gives the following theorem:

Theorem 1: Linearity of Heat Equation
If both u_1 and u_2 solve the heat equation, then so does:

$$c_1 u_1 + c_2 u_2$$

for all constants c_1 and c_2.

We can extend Theorem 1 by deriving an infinite set of solutions $\{u_1, u_2, \dots\}$, all of which solve the heat equation. In this case, by repeated application of Theorem 1 we find:

$$\sum_{k=1}^{\infty} c_k u_k$$

is also a solution to the heat equation, where $c_1, c_2 \dots$ are constants. This is called the principle of superposition (Zill & Cullen, 1992, p. 768).

3. Boundary-Value Problems

3.1 Introduction

We have discussed the mathematical meaning of the heat equation. Now we shall demonstrate its application toward solving physical problems. A *boundary-value problem* requires the finding of a $u(x,t)$ satisfying both *initial conditions:*

$$u(x,0) = f(x), \qquad x \in (0,L)$$

specifying the state of the system at $t = 0$, and also conditions describing the behaviour of the system's endpoints. These latter conditions are known as *boundary conditions*. Together they determine a unique solution, allowing us to predict heat flow within the rod.

3.2 Boundary Conditions

Various boundary conditions are possible for the heat equation. In this essay, we shall explore three important conditions for which the heat equation can be solved using Fourier series.

Let x_0 be an endpoint of a rod lying on the x-axis. The Dirichlet condition is:

$$u(x_0,t) = a$$

This states that the endpoint is held at a fixed temperature a. For instance, if the end of a rod is held in an ice bath of temperature 0, then the temperature of the rod at the endpoint will also be at zero temperature.

The Neumann condition:

$$u_x(x_0,t) = a$$

When we derived the heat equation, we used the empirical law:

$$Q_t = -KAu_x$$

From this, we can conclude that if $u_x(x_0,t) = a$, then:

$$Q_t = -KAa$$

The Neumann condition is thus equivalent to stating that heat flow through the endpoint occurs at a constant rate. If $a = 0$, no heat flow occurs; this represents a rod where the endpoint is insulated.

We shall examine a rod lying between $x = 0$ and $x = L$. Using these two conditions, we have the following boundary-value problems, which we must solve for the initial condition $u(x,0) = f(x)$:

1. Dirichlet Problem:

$$u(0,t) = a$$
$$u(L,t) = b$$

2. Neumann Problem:

$$u_x(0,t) = r$$
$$u_x(L,t) = r$$

We can add to this a third possible boundary value problem. Let us say we have a ring, connecting to itself at the endpoints. This problem is of both theoretic and historic interest, as it to this problem that Fourier first applied his Fourier series, and was also the first experimentally verified solution (Carslaw & Jaeger, 1959, p. 160):

3. Fourier's Ring:

$$u(0,t) = u(L,t)$$
$$u_x(0,t) = u_x(L,t)$$

3.3 Linear and Nonlinear Boundaries

Before we can solve the boundary-value problems posed previously, we need some background theory. We shall begin with a definition:

> **Definition:** A boundary condition is said to be *linear* if, for any functions $u(x,t)$ and $v(x,t)$ satisfying the boundary conditions, all linear combinations:
> $$c_1 u(x,t) + c_2 v(x,t), \qquad c_1, c_2 \in \mathbb{R}$$
> also satisfy the boundary conditions. The boundary condition is otherwise *nonlinear*.

The condition:

$$u(0,t) = 0$$

is linear, because, if u_1 and u_2 both solve the boundary condition, then:

$$c_1 u_1(0,t) + c_2 u_2(0,t) = 0$$

and thus so does $c_1 u_1 + c_2 u_2$. However, if we study the more general condition:

$$u(0,t) = a$$

then if u_1 and u_2 both solve the boundary conditions:

$$c_1 u_1(0,t) + c_2 u_2(0,t) = c_1 a + c_2 a = a(c_1 + c_2) \neq 0 \; \forall \, c_1, c_2 \in \mathbb{R}$$

Thus this condition is nonlinear.

From the definition of a linear boundary condition, if u_1 satisfies a linear boundary condition, then so must:

$$u_1 - u_1 = 0$$

Therefore, $u(x,t) = 0$ satisfies all linear boundary conditions.

We can use this to classify the boundary conditions stated in the previous section. Table 1 shows the linear and nonlinear cases of these conditions, and serves as a summary of this section.

Boundary –Value Problem	Linear Case	Nonlinear Case
Dirichlet Problem	$u(0,t) = 0$ $u(L,t) = 0$	$u(0,t) = a$ $u(L,t) = b$
Neumann Problem	$u_x(0,t) = 0$ $u_x(L,t) = 0$	$u_x(0,t) = r$ $u_x(L,t) = r$
Fourier's Ring	$u(0,t) = u(L,t)$ $u_x(0,t) = u_x(L,t)$	N/A

Table 1: Linear and nonlinear boundary conditions

4. Fourier Series

4.1 Introduction

Joseph Fourier (1768-1830) was the first mathematician to systematically solve the problems we described in the previous section. He did so by combining two different methods: Fourier series, introduced in this section, and Separation of Variables (also known as Fourier's Method) developed in part 5.

The idea of expanding a function into a series of form:

$$\sum_{n=0}^{\infty}[a_n \cos(nx) + b_n \sin(nx)]$$

was used by Bernoulli and Euler in the mid 18th century to explore string vibrations. However, Fourier was the first to suggest that all functions can be written in this form. Opposed by contemporary mathematicians because of his lack of rigour, it would take the work of Dirichlet and Riemann to prove the validity of his belief. Fourier series are now used not only for solving PDEs, but are essential in communications theory, and speech and image processing (James, 1993, pp. 283-285). The next three sections are based off both the previously cited book and also (Brand, 1955, pp. 511-539), which contains a more difficult but more rigorous chapter on the subject.

4.2 Fourier Series

Two functions $f(x)$ and $g(x)$ are said to be orthogonal on $[a, b]$ if:

$$\int_a^b f(x)g(x)\,dx = 0$$

If all of the functions in a set $\{\phi_n(x)\}$ are mutually orthogonal, then we say that they form an orthogonal set.

The set:

$$\mathcal{F} = \left\{\frac{1}{2}, \cos\left(\frac{\pi x}{p}\right), \cos\left(\frac{2\pi x}{p}\right), \dots, \sin\left(\frac{\pi x}{p}\right), \sin\left(\frac{2\pi x}{p}\right) \dots\right\}$$

is an orthogonal set on the interval $[-p, p]$. The proof is simple but long, and has been left to Appendix A, as are the evaluations of many of the integrals which follow.

A Fourier series expands a function $f(x)$ in terms of the elements of $\mathcal{F}$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right)\right]$$

where $x \in [-p, p]$. The series of constants a_n and b_n are known as the Fourier coefficients of $f(x)$. We can use the orthogonality of $\mathcal{F}$ to calculate these coefficients, by taking the integral:

$$\int_{-p}^{p} f(x)g(x)\ dx$$

where $g(x) = \cos\left(\frac{k\pi x}{p}\right), n \in \mathbb{Z}$:

$$\int_{-p}^{p} \cos\left(\frac{k\pi x}{p}\right) f(x)dx = \int_{-p}^{p} \cos\left(\frac{k\pi x}{p}\right)\left(\frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right)\right]\right) dx$$

$$= a_0 \int_{-p}^{p} \cos\left(\frac{k\pi x}{p}\right)\ dx$$

$$+ \sum_{n=1}^{\infty}\left[a_n \int_{-p}^{p} \cos\left(\frac{n\pi x}{p}\right) \cos\left(\frac{k\pi x}{p}\right)\ dx\ + b_n \int_{-p}^{p} \cos\left(\frac{n\pi x}{p}\right) \sin\left(\frac{k\pi x}{p}\right)\ dx\right]$$

$$= a_k \int_{-p}^{p} \cos^2\left(\frac{kk\pi x}{p}\right)\ dx\ = a_k p$$

$$\therefore a_k = \frac{1}{p}\int_{-p}^{p} \cos\left(\frac{k\pi x}{p}\right) f(x)dx$$

Repeating this with $g(x) = \sin\left(\frac{k\pi x}{p}\right)$ gives:

$$b_k = \frac{1}{p}\int_{-p}^{p} \sin\left(\frac{k\pi x}{p}\right) f(x)dx$$

We summarise the results of this discussion as a theorem:

Theorem 2: Calculating Fourier Coefficients
Let:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right)\right]$$

Then the coefficients must satisfy:

$$a_k = \frac{1}{p}\int_{-p}^{p} \cos\left(\frac{k\pi x}{p}\right) f(x)\ dx$$
$$b_k = \frac{1}{p}\int_{-p}^{p} \sin\left(\frac{k\pi x}{p}\right) f(x)\ dx$$

Example

We will find the Fourier coefficients of the function $f(x) = x$ on the interval $(-1,1)$. Using Theorem 2, we find that:

$$a_n = \int_{-1}^{1} x\cos(n\pi x)\ \ dx$$

$$b_n = \int_{-1}^{1} x \sin(n\pi x)\ dx$$

Using integration by parts:

$$a_n = \int_{-1}^{1} x \cos(n\pi x)\ dx = 0$$
$$b_n = \int_{-1}^{1} x \sin(n\pi x)\ dx = \frac{2}{n\pi}(-1)^{n+1}, n \in \mathbb{Z}^+$$

Therefore, the Fourier series is given by:

$$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi x) = \frac{2}{\pi}\left(\sin(\pi x) - \frac{\sin(2\pi x)}{2} + \frac{\sin(3\pi x)}{3} - \cdots\right)$$

By plotting the partial sums of this series, we can see that this does indeed converge to $f(x)$ on the interval $(-1,1)$ shown in *Figure 1*.

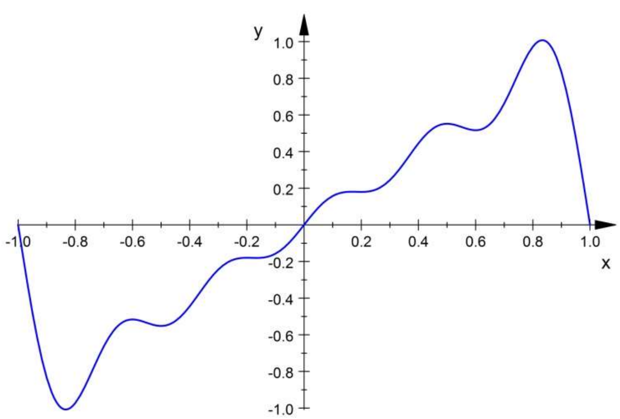
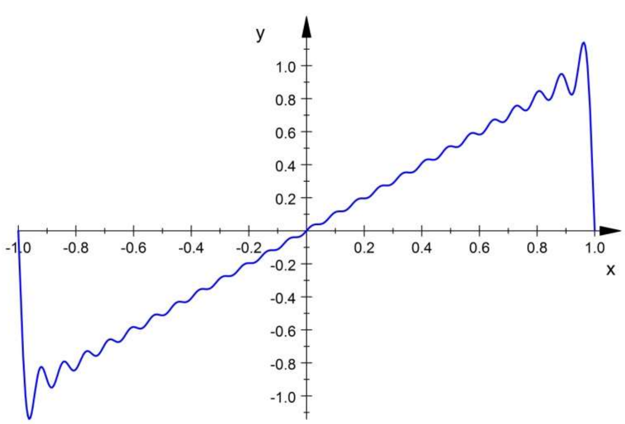
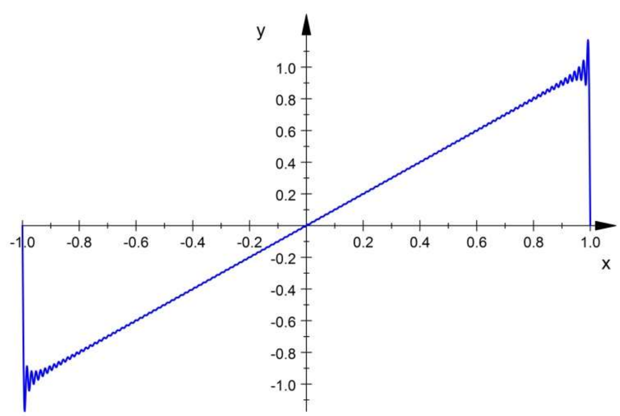


Figure 1: Partial Sums for $\mathbf{n} = \mathbf{5}$ (left), $\mathbf{n} = \mathbf{25}$ (bottom left), and $\mathbf{n} = \mathbf{125}$ (directly below)





4.3 Properties of Fourier Series

In the previous section we showed that, if a function has a Fourier series, then we can calculate the coefficients using Theorem 3. We aim to show that a function can be represented by a Fourier series; that is, our calculated Fourier series actually converges to the function. Our next theorem will place sufficient conditions to guarantee convergence; we state it without proof.

> **Theorem 3: Convergence Theorem**
> Let $f(x)$ be bounded with a finite number of maxima and minima on an interval $[-p, p]$. Its Fourier series then converges at every continuous point.

These conditions are known as the Dirichlet conditions. When they hold, we can write:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right) \right]$$

without concern for convergence. For the physical problems we are dealing with, these conditions should hold, thus, convergence is not an issue.

Fourier series can be added and subtracted, integrated and differentiated to form new Fourier series. This allows us to evaluate the Fourier series of a function without having to evaluate the integrals of Theorem 2. However, if a Fourier series is differentiated, the new series may not necessarily converge.

4.4 Half-Range Expansions

A function is odd if:

$$f(-x) = -f(x)$$

and is even if:

$$f(-x) = f(x)$$

The function $\sin(kx)$ is odd, and $\cos(kx)$ is even. We can simplify our evaluation of Fourier series by using the following properties of odd and even functions (James, 1993, p. 298):

1. The product of two even functions is even
2. The product of two odd functions is even
3. The product of an even and an odd function is odd
4. If $f(x)$ is odd then $\int_{-p}^{p} f(x)\, dx = 0$

These properties can all easily be derived from the definition of odd and even functions. Using these properties, we find if $f(x)$ is even, then:

$$\int_{-p}^{p} f(x) \sin\left(\frac{n\pi x}{p}\right) dx = 0$$

and thus the Fourier series of $f(x)$ is of the form:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right)$$

Likewise, if $f(x)$ is odd, then:

$$\int_{-p}^{p} f(x) \cos\left(\frac{n\pi x}{p}\right) dx = 0$$

in which case:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right)$$

When solving the heat equation, we are not interested in the interval $(-p, p)$ but rather the interval $(0, L)$. Using our Fourier series, we shall derive three *half-range* series. Say we have a function $f(x)$ defined on $(0, L)$.

If we let $f(-x) = f(x)\ \forall\ x \in (0, L)$, then we have created an even function on the interval $(-L, L)$. Then $f(x)$ has a Fourier series comprised of cosines, and therefore, from Theorem 2:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \qquad x \in (0, L)$$

where

$$a_n = \frac{2}{L}\int_0^L \cos\left(\frac{n\pi x}{L}\right) f(x) dx$$

This is a half-range cosine expansion.

If instead we defined $f(-x) = -f(x)\ \forall\ x \in (0, L)$, then we have created an odd function. The Fourier series is thus given by:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \qquad x \in (0, L)$$

where

$$b_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x) dx$$

and this is a half-range sine expansion.

Rather than expanding $f(x)$ as a sine or cosine series, we can instead expanding it as a half-range Fourier series by letting $f(x - L) = f(x) \forall\ x \in (0, L)$. As it is neither odd nor even, its Fourier series is:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2n\pi x}{L}\right) + b_n \sin\left(\frac{2n\pi x}{L}\right)\right], \qquad x \in (0, L)$$

where:

$$a_n = \frac{2}{L}\int_0^L \cos\left(\frac{2n\pi x}{L}\right) f(x)\, dx$$
$$b_n = \frac{2}{L}\int_0^L \sin\left(\frac{2n\pi x}{L}\right) f(x)\, dx$$

Example

We shall find the half-range expansions of $f(x) = x^2,\ \ 0 < x < 1$.

Cosine Series

We must find:

$$a_n = 2\int_0^1 x^2 \cos(n\pi x)\, dx$$

Using integration by parts, we obtain:

$$a_0 = \frac{2}{3}$$
$$a_n = \frac{4(-1)^n}{n^2\pi^2}, \qquad k \in \mathbb{Z}^+$$

and thus:

$$x^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(n\pi x), \qquad 0 < x < 1$$

Sine Series

We must calculate:

$$b_k = 2\int_0^1 x^2 \sin(n\pi x)\, dx$$

Integration by parts gives:

$$b_n = \frac{2(-1)^{n+1}}{n\pi} + \frac{4((-1)^n - 1)}{n^3\pi^3}$$

Therefore:

$$x^2 = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}}{n} + \frac{2((-1)^n - 1)}{n^3\pi^2}\right)\sin(n\pi x), \qquad 0 < x < 1$$

Fourier Series

We calculate:

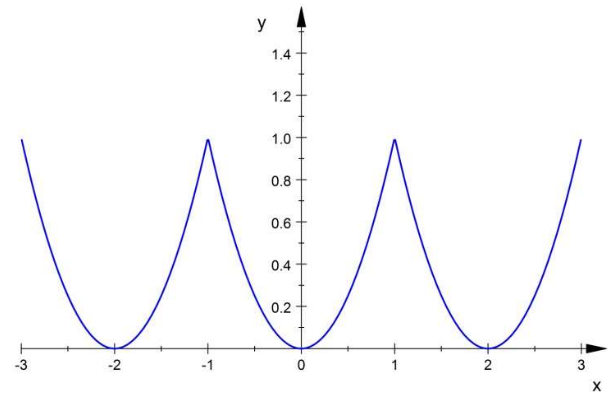
$$a_0 = 2\int_0^1 x^2 dx = \frac{2}{3}$$
$$a_n = 2\int_0^1 x^2 \cos(2n\pi x)\, dx = \frac{1}{n^2\pi^2}$$
$$b_n = 2\int_0^1 x^2 \sin(2n\pi x)\, dx = -\frac{1}{n\pi}$$

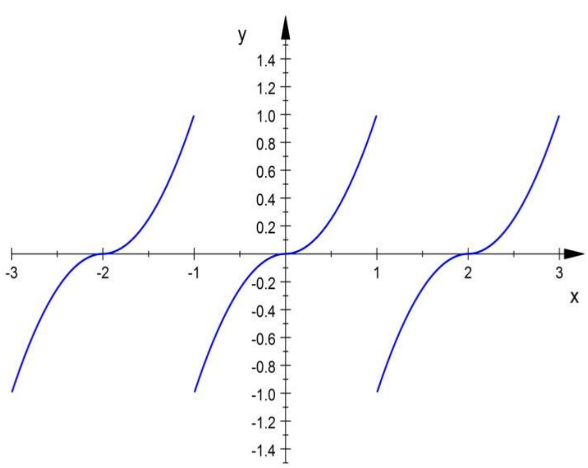
giving the Fourier series:

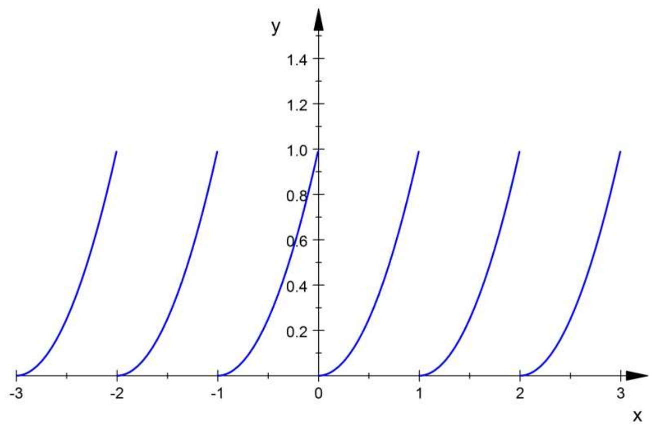
$$x^2 = \frac{1}{3} + \frac{1}{\pi}\sum_{k=1}^{\infty}\left[\frac{1}{n^2\pi}\cos(2n\pi x) - \frac{1}{n}\sin(2n\pi x)\right], \qquad 0 < x < 1$$

We can plot the functions that these three series converge, as shown in *Figure 2*.

Figure 2: Half-Range cosine expansion(left), sine expansion (Bottom Left), and Fourier series (Below)







5. General Solutions

5.1 Separation of Variables

Before solving boundary-value problems, we must first find a general set of solutions to the heat equation. This can be done using a technique known as Separation of Variables.

Assume that:

$$u(x,t) = X(x)T(t)$$

By substituting $X(x)T(x)$ into the heat equation, we obtain:

$$\frac{\partial\big(X(x)T(t)\big)}{\partial t} = k\frac{\partial^2\big(X(x)T(t)\big)}{\partial x^2}$$

Thus:

$$X(x)T'(t) = kX''(x)T(t)$$
$$\therefore \frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)}$$

This is only possible is if both functions are constant, thus:

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = C$$

We have thus separated the PDE into the two ordinary differential equations:

$$CX(x) = X''(x)$$
$$CkT(t) = T'(t)$$

There are three possibilities:

Case 1: $C > 0$

For algebraic convenience we write $C = \lambda^2$.Therefore:

$$X''(x) = \lambda^2 X(x)$$
$$T'(t) = k\lambda^2 T(t)$$

These differential equations are solved by:

$$X(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$$
$$T(t) = c_3 e^{k\lambda^2 t}$$

where c_1, c_2 and c_3 are constants. Multiplying these expressions and absorbing the constant c_3 gives a general solution of the form:

$$\alpha_\lambda(x,t) = X(x)T(t) = e^{k\lambda^2 t}\big(c_1 e^{\lambda x} + c_2 e^{-\lambda x}\big)$$

Case 2: $C = 0$ or $1/C = 0$

If the latter is true then this results in the trivial solution $X(x) = 0$, $T(t) = 0$. Assuming instead the former:

$$X''(x) = 0$$
$$T'(t) = 0$$

Thus:

$$X(x) = c_1 x + c_2$$
$$T(t) = c_3$$

Multiplying these together and absorbing c_3 gives:

$$\beta(x,t) = X(x)T(t) = c_1 x + c_2$$

Case 3: $C < 0$

For algebraic convenience we can write $C = -\lambda^2$. Therefore:

$$X''(x) = -\lambda^2 X(x)$$
$$T'(t) = -k\ {}^2 T(t)$$

These equations must thus be of the form:

$$X(x) = c_1 \sin(\lambda x) + c_2 \cos(\lambda x)$$
$$T(t) = c_3 e^{-\lambda^2 kt}$$

Multiplying these together and absorbing c_3 gives:

$$\gamma_\lambda(x,t) = e^{-\lambda^2 kt}(c_1 \sin(\lambda x) + c_2 \cos(\lambda x))$$

5.2 General Solutions of the Heat Equation

Using separation of variables, we found three solutions to the heat equation:

$$\alpha_\lambda(x,t) = e^{\lambda^2 kt}\left(c_1 e^{\lambda x} + c_2 e^{-\lambda x}\right)$$
$$\beta(x,t) = c_1 x + c_2$$
$$\gamma_\lambda(x,t) = e^{-\lambda^2 kt}(c_1 \sin(\lambda x) + c_2 \cos(\lambda x))$$

These three different solutions are the general solutions of the heat equation, in the sense that any solution to the heat equation can be derived from these solutions using the principle of superposition.

The first solution α_λ is unstable, as:

$$\lim_{t\to\infty} \alpha_\lambda(x,t) = \lim_{t\to\infty} e^{\lambda^2 kt}\left(c_1 e^{\lambda x} + c_2 e^{-\lambda x}\right)$$

becomes infinitely large for any value of x.

Both the second and third solution are stable. The second solution β is a stationary solution, as it does not vary with time, and the third solution γ_λ decays exponentially over time for any value of x:

$$\lim_{t\to\infty} \gamma_\lambda(x,t) = \lim_{t\to\infty} e^{-\lambda^2 kt}(c_1 \sin(\lambda x) + c_2 \cos(\lambda x)) = 0$$

As the boundary-value problems we shall solve always have stable solutions, we shall find that the solutions to these problems are comprised of terms of the form β and γ_λ.

6. Fourier's Ring

6.1 General Solution

The first boundary-value problem we shall solve shall, like Fourier, is that of a ring. The boundary conditions are:

$$u(0,t) = u(L,t)$$
$$u_x(0,t) = u_x(L,t)$$

We shall attempt to find functions of the form α_λ, β, and γ_λ satisfying these conditions. If:

$$\alpha_\lambda(0,t) = \alpha_\lambda(L,t)$$
$$(\alpha_\lambda)_x(0,t) = (\alpha_\lambda)_x(L,t)$$

we obtain:

$$e^{\lambda^2 kt}(c_1 + c_2) = e^{\lambda^2 kt}\left(c_1 e^{\lambda L} + c_2 e^{-\lambda L}\right)$$
$$e^{\lambda^2 kt}\lambda(c_1 - c_2) = e^{\lambda^2 kt} 2\lambda\left(c_1 e^{\lambda L} - c_2 e^{-\lambda L}\right)$$

This simplifies to:

$$c_1 + c_2 = c_1 e^{\lambda L} + c_2 e^{-\lambda L}$$
$$c_1 - c_2 = c_1 e^{\lambda L} - c_2 e^{-\lambda}$$

Adding and subtracting these two equations gives:

$$2c_1 = 2c_1 e^{\lambda L}$$
$$2c_2 = 2c_2 e^{-\lambda L}$$

$$\therefore e^{\lambda L} = e^{-\lambda L} = 1$$

This is only possible if $\lambda L = 0$. As L is not 0, we conclude $\lambda = 0$, and the only solution is therefore the constant:

$$\alpha_0(x,t) = c_1$$

Repeating this procedure with β gives us the same result. We are therefore left with $\gamma_\lambda(x,t)$:

$$\gamma_\lambda(0,t) = \gamma_\lambda(L,t)$$
$$(\gamma_\lambda)_x(0,t) = (\gamma_\lambda)_x(L,t)$$

Therefore:

$$e^{-\lambda^2 kt}(c_1\sin(\lambda 0) + c_2\cos(\lambda 0)) = e^{-\lambda^2 kt}(c_1\sin(\lambda L) + c_2\cos(\lambda L))$$
$$e^{-\lambda^2 kt}\lambda(c_1\cos(\lambda 0) - c_2\sin(\lambda 0)) = e^{-\lambda^2 kt}\lambda(c_1\cos(\lambda L) - c_2\sin(\lambda L))$$

Thus:

$$c_2 = c_1 \sin(\lambda L) + c_2 \cos(\lambda L)$$
$$c_1 = c_1 \cos(\lambda L) - c_2 \sin(\lambda L)$$

$$\therefore c_1 \sin(\lambda L) + c_2(\cos(\lambda L) - 1) = 0$$
$$\therefore c_1(\cos(\lambda L) - 1) - c_2 \sin(\lambda L) = 0$$

We can write this in matrix form as:

$$\begin{pmatrix} \sin(\lambda L) & \cos(\lambda L) - 1 \\ \cos(\lambda L) - 1 & -\sin(\lambda L) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

There are nonzero solutions only if the matrix is singular:

$$\begin{vmatrix} \sin(\lambda L) & \cos(\lambda L) - 1 \\ \cos(\lambda L) - 1 & -\sin(\lambda L) \end{vmatrix} = 0$$

$$\therefore -\sin^2(\lambda L) - (\cos^2(\lambda L) - 2\cos(\lambda L) + 1) = 0$$

$$\therefore \cos(\lambda L) = 1$$

$$\therefore \lambda L = 2\pi n, \qquad n \in \mathbb{N}$$

Letting:

$$\lambda_n = \frac{2\pi n}{L}, \qquad n \in \mathbb{N}$$

we find any function:

$$u_n(x,t) = e^{\frac{-4k\pi^2 n^2}{L^2}t}\left(a_n \cos\left(\frac{2n\pi x}{L}\right) + b_n \sin\left(\frac{2n\pi x}{L}\right)\right)$$

solves Fourier's ring. By the principle of superposition, any function of the form:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} e^{\frac{-4k\pi^2 n^2}{L^2}t}\left(a_n \cos\left(\frac{2n\pi x}{L}\right) + b_n \sin\left(\frac{2n\pi x}{L}\right)\right)$$

solves Fourier's Ring.

6.2 Applying Initial Conditions

We now need to solve the problem for the initial condition:

$$u(x,0) = f(x)$$

Substituting $t = 0$ into our previously found series for $u(x,t)$, we find:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2n\pi}{L}\right) + b_n \sin\left(\frac{2n\pi x}{L}\right)\right]$$

This is a Fourier half-series. From our work in section 4.4, we can calculate the coefficients a_n and b_n to establish the solution:

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} e^{\frac{-4k\pi^2 n^2}{L^2}t}\left(a_n \cos\left(\frac{2n\pi x}{L}\right) + b_n \sin\left(\frac{2n\pi x}{L}\right)\right)$$

satisfying both the boundary conditions and the initial conditions. We can summarise the results of this section as a theorem:

Theorem 4: Solving Fourier's Ring

Suppose we have a ring with an initial temperature distribution given by:

$$u(x,0) = f(x), \qquad 0 < x < L$$

As heat flows through the ring, the temperature at a given point and time will be given by:

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} e^{\frac{-4k\pi^2 n^2}{L^2}t}\left(a_n \cos\left(\frac{2n\pi x}{L}\right) + b_n \sin\left(\frac{2n\pi x}{L}\right)\right)$$

where:

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{2n\pi x}{L}\right)\,dx$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{2n\pi x}{L}\right)\,dx$$

and where k is the thermal diffusivity of the ring.

6.3 Example

In section 4.4, we showed that:

$$x^2 = \frac{1}{3} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{n^2\pi}\cos(2n\pi) - \frac{1}{n}\sin(2n\pi)\right], \qquad 0 < x < 1$$

Thus, if we have a ring with an initial temperature distribution of:

$$u(x,0) = x^2$$

then the temperature distribution is given by:

$$u(x,t) = \frac{1}{3} + \frac{1}{\pi}\sum_{n=1}^{\infty} e^{-4kn^2\pi^2 t}\left(\frac{1}{n^2\pi}\cos(2n\pi) - \frac{1}{n}\sin(2n\pi)\right), \qquad 0 < x < 1, \;\; t \geq 0$$

Figure 3 shows the evolution of this system over time when $k = 100$. This reflects what intuitively we would expect to happen.

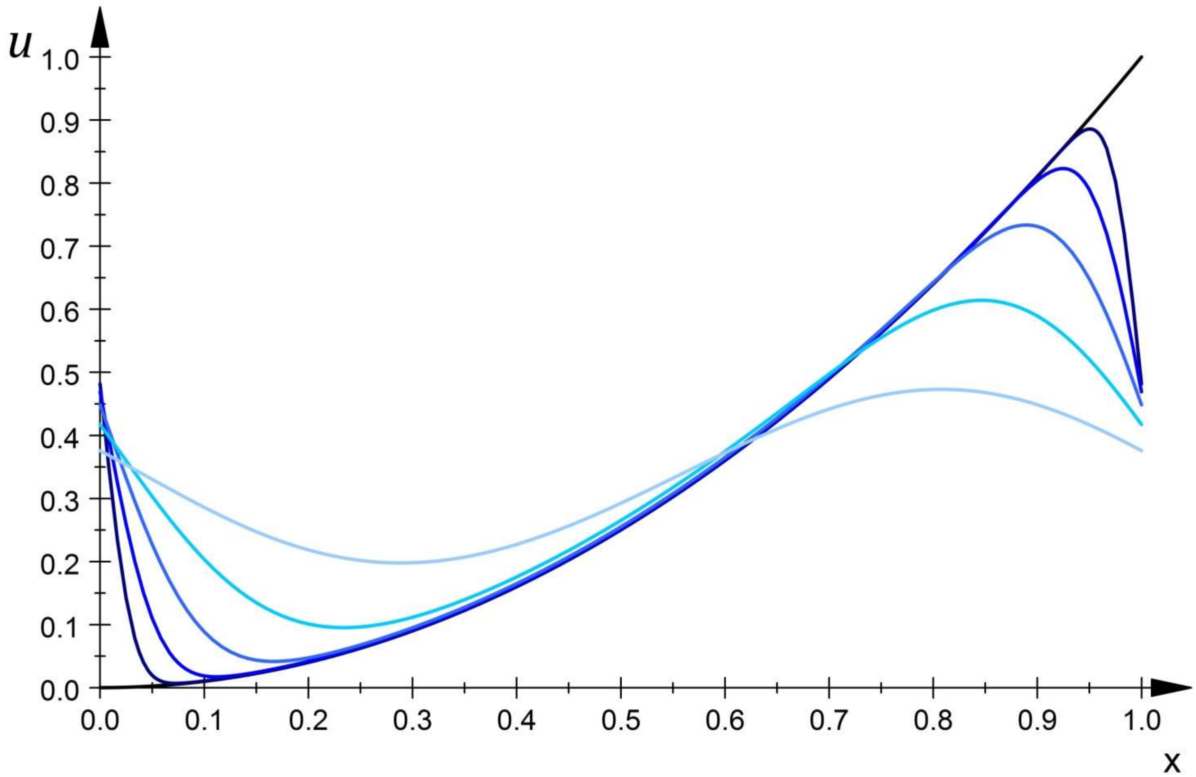


Figure 3: The temperature distribution at, from darkest to lightest:

$$\boldsymbol{t = 0,\quad t = \frac{1}{9},\quad t = \frac{1}{3},\quad t = 1\,,\quad t = 3,\quad t = 9}$$

7. Dirichlet Problem

7.1 Linear Dirichlet Problem

We shall begin by solving the linear Dirichlet Problem:

$$u(0,t) = 0$$
$$u(L,t) = 0$$

Substituting $u(x,t) = \alpha_\lambda$, we find:

$$u(0,t) = e^{k\lambda^2 t}(c_1 + c_2) = 0$$
$$u(L,t) = e^{k\lambda^2 t}\left(c_1 e^{\lambda L} + c_2 e^{-\lambda L}\right) = 0$$

As $e^{k\lambda^2 t} \neq 0$:

$$\therefore c_1 + c_2 = 0$$
$$\therefore c_1 e^{\lambda L} + c_2 e^{-\lambda L} = 0$$

In matrix form this is:

$$\begin{pmatrix} 1 & 1 \\ e^{\lambda L} & e^{-\lambda L} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This has nonzero solutions only if the matrix is singular:

$$\therefore \begin{vmatrix} 1 & 1 \\ e^{\lambda L} & e^{-\lambda} \end{vmatrix} = e^{-\lambda} \ - e^{\lambda L} = 0$$

$$\therefore \lambda = 0$$

If $\lambda = 0$, then $c_1 = -c_2$, with trivial solution:

$$\alpha_\lambda = 0$$

This also applies to the second condition. However, by substituting into γ_λ, we find:

$$\gamma_\lambda(0,t) = e^{-k\lambda^2 t}(c_1 \sin(\lambda 0) + c_2 \cos(\lambda 0)) = e^{-k\lambda^2 t} c_2 = 0$$
$$\gamma_\lambda(L,t) = e^{-k\lambda^2 t}(c_1 \sin(\lambda L) + c_2 \cos(\lambda L))$$

Therefore:

$$c_2 = 0$$

$$\therefore \gamma_\lambda(L,t) = e^{-k\lambda^2 t} c_1 \sin(\lambda L) = 0$$

If $c_1 = 0$ then we have the trivial solution $\gamma_0(L,t) = 0$. Instead, we shall let $\sin(\lambda L) = 0$.

$$\therefore \lambda L = n \ , \qquad n \in \mathbb{N}$$

Let:

$$\therefore \lambda_n = \frac{n\pi}{L}, \qquad n \in \mathbb{N}$$

We have found a series of functions

$$u_n(x,t) = e^{-k\lambda_n{}^2 t}\sin(\lambda_n x)$$

that solve the linear Dirichlet problem. By the principle of superimposition, any function of the form:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} b_n e^{-\frac{kn^2\pi^2}{L^2}t}\sin\left(\frac{n\pi}{L}x\right)$$

is therefore a solution.

7.2 Nonlinear Dirichlet Problem

We shall now solve the more general Dirichlet problem:

$$u(0,t) = a$$
$$u(L,t) = b$$

We can find a solution of form $\beta(x,t)$ that solves these conditions. Substituting:

$$\beta(0,t) = c_1 0 + c_2 = c_2 = a$$
$$\beta(L,t) = c_1 L + c_2 = b$$

Solving for c_1 and c_2 gives us:

$$\beta(x,t) = \frac{b-a}{L}x + a$$

as a solution to the boundary-value problem. If we add this to the solution found in the previous section:

$$u(x,t) = a + \frac{b-a}{L}x + \sum_{n=1}^{\infty} b_n e^{-\frac{kn^2\pi^2}{L^2}t}\sin\left(\frac{n\pi}{L}x\right)$$

then this also solves the boundary conditions and the heat equation. This is the general solution to the general Dirichlet problem. Notice that, by taking the limit as $t \to \infty$:

$$\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty} a + \frac{b-a}{L}x + \sum_{n=1}^{\infty} b_n e^{-\frac{kn^2\pi^2}{L^2}t}\sin\left(\frac{n\pi}{L}x\right) = a + \frac{b-a}{L}x$$

and thus, regardless of the initial conditions, the steady-state solution is always the linear function:

$$\lim_{t\to\infty} u(x,t) = a + \frac{b-a}{L}x$$

7.3 Applying the Initial Conditions

We shall now solve our problem for the initial-value conditions

$$u(x,0) = f(x)$$

By substituting $t = 0$ into the series for $u(x,t)$ found in the previous section, we find:

$$f(x) = a + \frac{b-a}{L}x + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$$

$$\therefore f(x) - a - \frac{b-a}{L}x = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$$

This is a half-range sine series. Using our work in section 4.4, we can thus expand the function $f(x) - a - \frac{b-a}{L}x$ into a half-range sine series in order to find the function:

$$u(x,t) = a + \frac{b-a}{L}x + \sum_{n=1}^{\infty} b_n e^{-\frac{kn^2\pi^2}{L^2}t} \sin\left(\frac{n\pi}{L}x\right)$$

satisfying both the boundary and initial conditions. We can summarise the results of this section as a theorem.

Theorem 5: Solving Dirichlet Problems

The solution to the Dirichlet Problem:

$$u(0,t) = a$$
$$u(0,t) = b$$

with initial condition:

$$u(x,0) = f(x), \qquad 0 < x < L$$

is the function:

$$u(x,t) = a + \frac{b-a}{L}x + \sum_{n=1}^{\infty} b_n e^{-\frac{kn^2\pi^2}{L^2}t} \sin\left(\frac{n\pi}{L}x\right)$$

where:

$$b_n = \frac{2}{L}\int_0^L \left(f(x) - a - \frac{b-a}{L}x\right) \sin\left(\frac{n\pi}{L}x\right)\, dx$$

7.4 Example

In section 4.2, we showed that:

$$x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi x) = \frac{2}{\pi}\left(\sin(\pi x) - \frac{\sin(2\pi x)}{2} + \frac{\sin(3\pi x)}{3} - \cdots\right)$$

when $x \in (-1,1)$. Thus, if we have initial condition:

$$u(x,0) = x$$

along with boundary conditions:

$$u(0,t) = u(1,t) = 0$$

then, from the previous section, we know that the temperature distribution over time will be given by:

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi x)\, e^{-kn^2\pi^2 t}$$

Figure 3 shows the evolution of the temperature distribution for this system when $k = 1$.

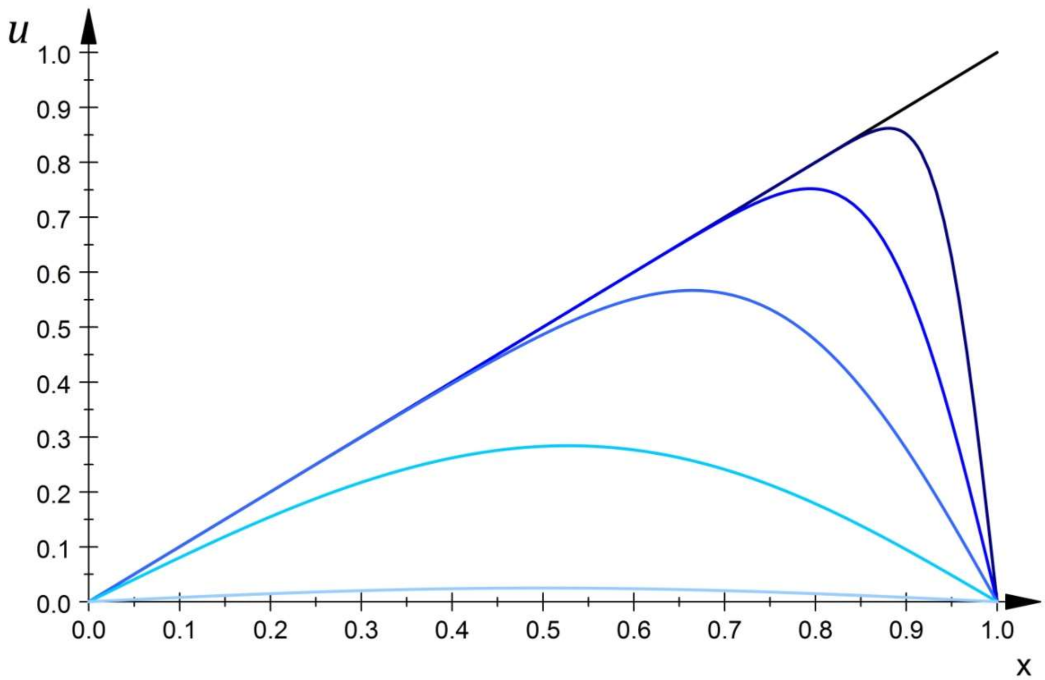


Figure 4: Temperature distributions, from darkest to lightest: $\boldsymbol{t = 0,\ t = 0.125,\ t = 0.5,\ t = 2,\ t = 8,\ t = 32}$.

Let us now impose the conditions:

$$u(x,0) = x$$

$$u(0,t) = 1$$
$$u(1,t) = 0$$

In this situation we find that the solution must be of the form

$$a + \frac{b-a}{L}x + \sum_{n=1}^{\infty} b_n e^{-\frac{kn^2\pi^2}{L^2}t} \sin\left(\frac{n\pi}{L}x\right) = 1 - x + \sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t} \sin(n\pi x)$$

where:

$$x = 1 - x + \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

$$\therefore 2x - 1 = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

Using section 4.4, we can calculate the coefficients using:

$$b_n = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right) f(x)dx = 2\int_0^1 \sin(n\pi x)\,(2x-1)dx$$

$$= \frac{2[2\sin(\pi n) - n\pi(\cos(n\pi)+1)]}{\pi^2 n^2}$$

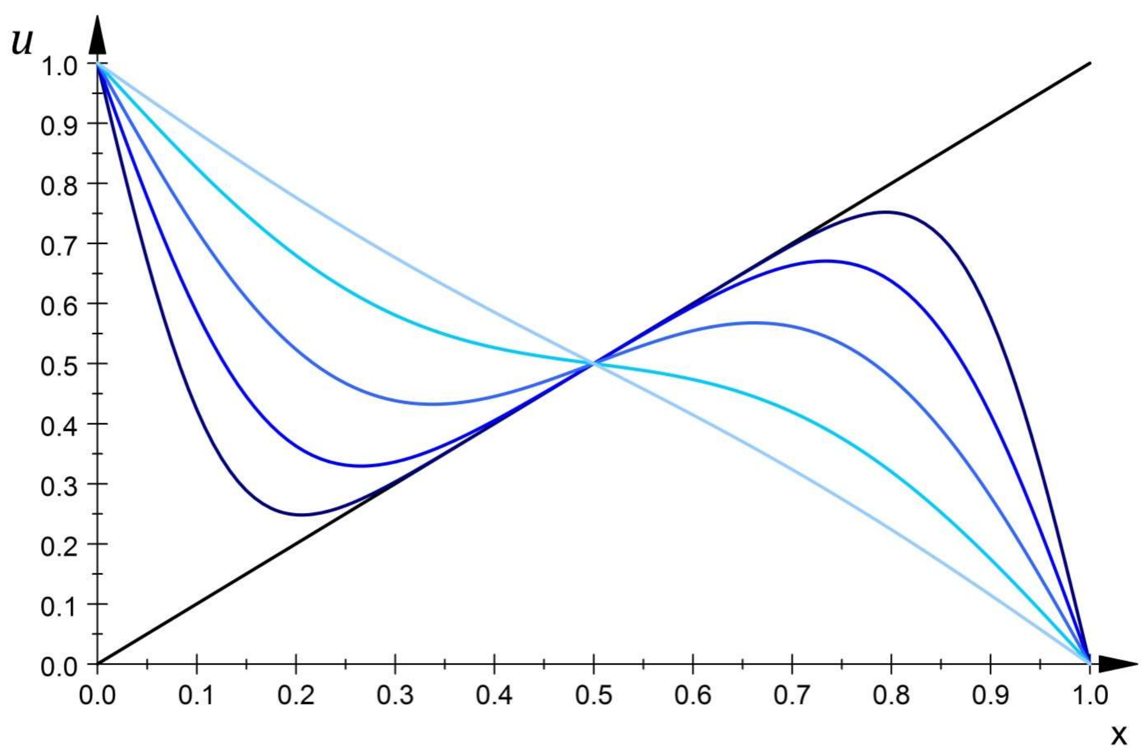
$$= \begin{cases} 0, & n \text{ is odd} \\ -\frac{4}{n\pi}, & n \text{ is even} \end{cases}$$

Therefore:

$$u(x,t) = 1 - x - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{e^{-4kn^2\pi^2 t}\sin(2n\pi x)}{2n}$$

Figure 5 shows the evolution of this temperature distribution for $k = 1$.

Figure 5: Temperature distributions, from darkest to lightest:
$\boldsymbol{t = 0,\ t = 0.5,\ t = 1,\ t = 2,\ t = 4,\ t = 8}$.



8. Neumann Problem

8.1 Linear Neumann Problem

Let:

$$u_x(0,t) = 0$$
$$u_x(L,t) = 0$$

The functions α_λ and β only solve these boundary conditions if they are constant. Once again, we can find a series of γ_λ that solve these conditions:

$$(\gamma_\lambda)_x(0,t) = e^{-k\lambda^2 t}(c_1\lambda\cos(\lambda 0) - c_2\lambda\sin(\lambda 0)) = c_1\lambda e^{-k\lambda^2 t} = 0$$
$$(\gamma_\lambda)_x(L,t) = e^{-k\lambda^2 t}(c_1\lambda\cos(\lambda L) - c_2\lambda\sin(\lambda L)) = 0$$

Thus:

$$c_1\lambda = 0$$
$$c_1\lambda\cos(\lambda L) + c_2\lambda\sin(\lambda L) = 0$$

Either $c_1 = 0$ or $\lambda = 0$. If $\lambda = 0$ then γ_0 is constant, otherwise:

$$c_1 = 0$$

If this is the case, then:

$$c_2\lambda\sin(\lambda L) = 0$$

If $c_2 = 0$, then we have the trivial solution $\gamma_\lambda = 0$. Instead, we shall let $\sin(\lambda L) = 0$.

$$\therefore \lambda L = n\pi, \qquad n \in \mathbb{N}$$

Let:

$$\therefore \lambda_n = \frac{n\pi}{L}, \qquad n \in \mathbb{N}$$

The functions:

$$u_n(x,t) = e^{-k{\lambda_n}^2 t}\cos(\lambda_n x)$$

therefore solve the boundary conditions. By the principle of superimposition, any function of the form:

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-\frac{kn^2\pi^2}{L^2}t}\cos\left(\frac{n\pi}{L}x\right)$$

solves the Neumann boundary conditions.

8.2 Nonlinear Neumann Problem

The nonlinear Neumann Problem is:

$$u_x(0,t) = r$$
$$u_x(L,t) = r$$

Letting $u(x,t) = c_1 x + c_2$ gives us the conditions:

$$c_1 = r$$
$$c_1 = r$$

The particular solution we seek is thus:

$$p(x,t) = rx$$

Any function of the form:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \frac{a_0}{2} + rx + \sum_{n=1}^{\infty} a_n e^{-\frac{kn^2\pi^2}{L^2}t} \cos\left(\frac{n\pi}{L}x\right)$$

solves the general Neumann conditions.

8.3 Applying Initial Conditions

By substituting $t = 0$ into our series solution, we find that, if:

$$f(x) = u(x,0)$$

then:

$$f(x) = \frac{a_0}{2} + rx + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right)$$

$$\therefore f(x) - rx = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right)$$

This is a half-range cosine series. By expanding the function $f(x) - rx$ into a half-range cosine series, we can then solve both the boundary conditions and the initial conditions with:

$$u(x,t) = \frac{a_0}{2} + rx + \sum_{n=1}^{\infty} a_n e^{-\frac{kn^2\pi^2}{L^2}t} \cos\left(\frac{n\pi}{L}x\right)$$

We can summarise the results of this section as a theorem.

Theorem 6: Solving Neumann Problems

The solution to the Neumann Problem:

$$u_x(0,t) = r$$
$$u_x(0,t) = r$$

with initial condition:

$$u(x,0) = f(x), \qquad 0 < x < L$$

is the function:

$$u(x,t) = \frac{a_0}{2} + rx + \sum_{n=1}^{\infty} a_n e^{-\frac{kn^2\pi^2}{L^2}t} \cos\left(\frac{n\pi}{L}x\right)$$

where:

$$a_n = \frac{2}{L}\int_0^L (f(x) - rx)\cos\left(\frac{n\pi}{L}x\right)\, dx$$

8.4 Example

We shall solve the linear Neumann problem with the initial condition:

$$u(x,0) = f(x) = \begin{cases} 1, & 0 < x < 0.5 \\ 0, & 0.5 < x < 1 \end{cases}$$

From the previous section, we know the solution to this problem is given by:

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-kn^2\pi^2 t}\cos(n\pi x)$$

where:

$$a_n = 2\int_0^1 \cos(n\pi x)\, f(x) dx$$

$$= 2\int_0^{0.5} \cos(n\pi x)\, dx$$

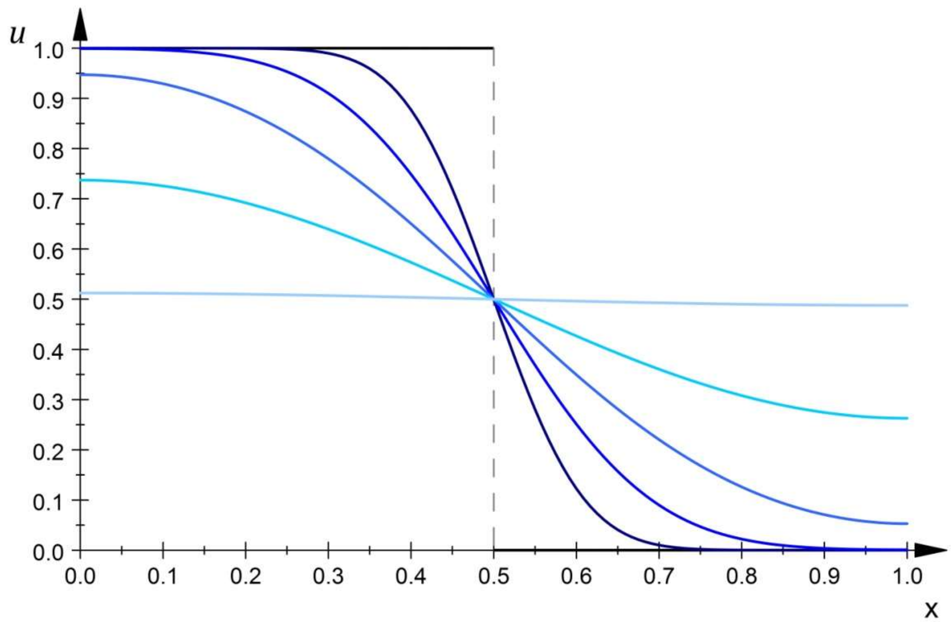
$$= \begin{cases} 1, & n = 0 \\ \dfrac{2}{n\pi}, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

Thus:

$$u(x,t) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=0}^{\infty} \frac{\cos\big((2n+1)\pi x\big)}{2n+1} e^{-k(2n+1)^2\pi^2 t}$$

Figure 6 shows the evolution of this function over time, for $k = 10$.

Figure 6: Temperature distributions, from darkest to lightest:
$\boldsymbol{t=0,\ t=3^{-3},\ t=3^{-2},\ t=3^{-1},\ t=1,\ t=3}$.



As a final example, let us study the Neumann problem with conditions:

$$u(x,0)=0$$

$$u_x(0,t)=1$$
$$u_x(L,t)=1$$

The general solution to this problem is:

$$u(x,t)=\frac{a_0}{2}+x+\sum_{n=1}^{\infty}a_n e^{-kn^2\pi^2 t}\cos(n\pi x)$$

Substituting $t=0$ gives us:

$$u(x,0)=0=\frac{a_0}{2}+x+\sum_{n=1}^{\infty}a_n\cos(n\pi x)$$

$$\therefore -x=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos(n\pi x)$$

Therefore:

$$a_n=-2\int_0^1\cos(n\pi x)\,x\,dx$$

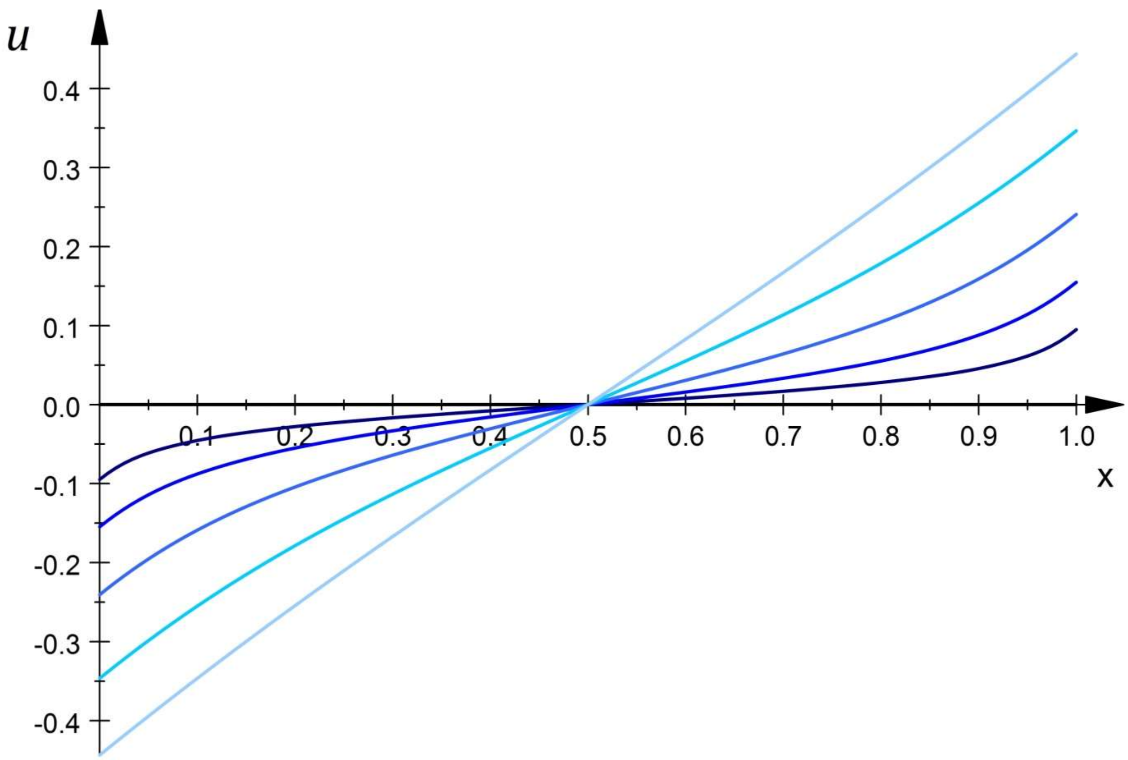
$$= \begin{cases} -1, & n = 0 \\ \dfrac{4}{\pi^2 n^2}, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

Thus:

$$u(x,t) = x - \frac{1}{2} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} e^{-kn^2\pi^2 t}\frac{\cos(n\pi x)}{n^2}$$

Figure 7 shows the evolution of this system, for $k = 10$.

Figure 7: Temperature distributions, from darkest to lightest: $\boldsymbol{t = 0,\ t = 0.125,\ t = 0.25,\ t = 0.5,\ t = 1,\ t = 2.}$



9. Conclusion

Our research question was:

> ***How Fourier series and the method of Separation of Variables can be used to solve the One-dimensional Heat Equation.***

First, we introduced and derived the one-dimensional heat equation, showing its application in physical problems. We then explored the Dirichlet, Neumann, and Fourier's Ring boundary-value problems. A summary of these problems is shown in *Table 2*.

Table 2: Summary of Boundary-Value Problems

	Dirichlet	Neumann	Fourier's Ring
Boundary Conditions	$u(0,t) = a$ $u(L,t) = b$	$u_x(0,t) = r$ $u_x(L,t) = r$	$u(0,t) = u(L,t)$ $u_x(0,t) = u_x(L,t)$
Linear Case	$u(0,t) = 0$ $u(L,t) = 0$	$u_x(0,t) = 0$ $u_x(L,t) = 0$	$u(0,t) = u(L,t)$ $u_x(0,t) = u_x(L,t)$
Steady State Solution	$a + \frac{b-a}{L}x$	rx	N/A

The theory of Fourier series was then introduced, showing how to represent a function as an infinite sum of sines and cosines. Separation of variables was used to find a series solution for the boundary conditions, and we then applied Fourier series to solve for the initial conditions. This allowed us to solve all three boundary-value problems. Our results are shown in *Table* 3.

Table 3: Summary of Solutions to the Heat Equation

Series Solution	**Dirichlet**	$u(x,t) = a + \frac{b-a}{L}x + \sum_{n=1}^{\infty} b_n e^{-\frac{kn^2\pi^2}{L^2}t} \sin\left(\frac{n\pi}{L}x\right)$
	Neumann	$u(x,t) = \frac{a_0}{2} + rx + \sum_{n=1}^{\infty} a_n e^{-\frac{kn^2\pi^2}{L^2}t} \cos\left(\frac{n\pi}{L}x\right)$
	Fourier's Ring	$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} e^{\frac{-4k\pi^2n^2}{L^2}t}\left(a_n \cos\left(\frac{2n\pi x}{L}\right) + b_n \sin\left(\frac{2n\pi x}{L}\right)\right)$

Series at t = 0 and Fourier Expansion used	**Dirichlet**	$u(x,0) = a + \frac{b-a}{L}x + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$ Fourier Half-Range Sine Expansion
	Neumann	$u(x,0) = \frac{a_0}{2} + rx + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right)$ Fourier Half-Range Cosine Expansion
	Fourier's Ring	$u(x,0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} e^{\frac{-4k\pi^2 n^2}{L^2}t}\left(a_n \cos\left(\frac{2n\pi x}{L}\right) + b_n \sin\left(\frac{2n\pi x}{L}\right)\right)$ Fourier Half-Range Full Expansion
Coefficients	**Dirichlet**	$b_n = \frac{2}{L}\int_0^L \left(f(x) - a - \frac{b-a}{L}x\right) \sin\left(\frac{n\pi}{L}x\right)\, dx$
	Neumann	$a_n = \frac{2}{L}\int_0^L (f(x) - rx) \cos\left(\frac{n\pi}{L}x\right)\, dx$
	Fourier's Ring	$a_n = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{2n\pi x}{L}\right)\, dx$ $b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{2n\pi x}{L}\right)\, dx$

In this essay, we have explored the simplest cases of an important area of study. Our combined method of separation of variables and Fourier series is applicable to solving other one-dimensional PDE's, such as the wave equation. It can also be extended to using Fourier series based on more advanced orthogonal functions, such as Legendre and Bessel functions, allowing us to solve for more complicated boundary conditions. Higher dimensional analogues of the heat equation can also be solved with such methods.

Other solution techniques also exist. Laplace transforms and conformal mappings both provide analytic solutions to the equation. However, numerical methods are also becoming increasing important, particularly with the adsvent of high-speed computing. The curious reader is directed to the book *Conduction of Heat in Solids* by H. S. Carslaw and J. C. Jaeger, which is the classic encyclopaedic reference text on the topic, containing a compendium of solutions to the heat equation.

Appendix A: Calculation of Integrals

In this appendix, we shall calculate the various integrals used in the text.

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We shall now show that the set:

$$\mathcal{F} = \left\{\frac{1}{2}, \cos\left(\frac{\pi x}{p}\right), \cos\left(\frac{2\pi x}{p}\right), \dots, \sin\left(\frac{\pi x}{p}\right), \sin\left(\frac{2\pi x}{p}\right) \dots\right\}$$

is an orthogonal set. We will first prove this on the interval $[-\pi, \pi]$. As:

$$\sin(-ax)\cos(-bx) = -\sin(ax)\cos(bx)$$
$$\therefore \int_{-\pi}^{\pi} \sin(ax)\cos(bx)\,dx = \int_{0}^{\pi} \sin(ax)\cos(bx)\,dx + \int_{-\pi}^{0} \sin(ax)\cos(bx)\,dx$$
$$= \int_{0}^{\pi} \sin(ax)\cos(bx)\,dx - \int_{0}^{\pi} \sin(ax)\cos(bx)\,dx = 0$$

and thus $\sin(ax)$ and $\cos(bx)$ are orthogonal for any a and b. As this holds true when $b = 0$, $\sin(ax)$ is orthogonal to $1/2$. This is also true for $\cos(bx)$, as:

$$\int_{-\pi}^{\pi} \cos(bx)\,dx = \frac{\sin(b\pi)}{b} - \frac{\sin(-b\pi)}{b} = 0 \;\forall\, b \in \mathbb{Z}, b \neq 0$$

If $a \neq b$ and $a, b \in \mathbb{Z}^+$, then:

$$\int_{-\pi}^{\pi} \cos\big((a+b)x\big)\,dx = 0$$
$$\int_{-\pi}^{\pi} \cos\big((a-b)x\big)\,dx = 0$$

Using trigonometric identities, we can write:

$$\int_{-\pi}^{\pi} \cos\big((a+b)x\big)\,dx = \int_{-\pi}^{\pi} \cos(ax)\cos(bx) - \sin(ax)\sin(bx)\,dx$$
$$\int_{-\pi}^{\pi} \cos\big((a-b)x\big)\,dx = \int_{-\pi}^{\pi} \cos(ax)\cos(bx) + \sin(ax)\sin(bx)\,dx$$

By adding or subtracting these two equations we gain the following formulae:

$$\therefore \int_{-\pi}^{\pi} \cos(ax)\cos(bx)\,dx = 0$$
$$\therefore \int_{-\pi}^{\pi} \sin(ax)\sin(bx)\,dx = 0$$

The functions $\left\{1, \cos\left(\frac{k\pi x}{p}\right), \sin\left(\frac{k\pi x}{p}\right) : k \in \mathbb{Z}^+\right\}$ are therefore orthogonal on the interval $[-\pi, \pi]$.This can be generalised to any interval $[-p, p]$ by making the substitution $u = \frac{xp}{\pi}$:

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-p}^{p} f\left(\frac{u\pi}{p}\right)\frac{\pi}{p}\,du$$

If $\int_{-\pi}^{\pi} f(x)\,dx = 0$, it follows that:

$$\int_{-p}^{p} f\left(\frac{u\pi}{p}\right)du$$

The functions $\left\{1, \cos\left(\frac{k\pi x}{p}\right), \sin\left(\frac{k\pi x}{p}\right) : k \in \mathbb{Z}^+\right\}$ are therefore orthogonal on the interval $[-p, p]$.

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Integral 1

$$\int_{-1}^{1} x\cos(n\pi x)\ dx = \int_{0}^{1} x\cos(n\pi x)\ dx + \int_{-1}^{0} x\cos(n\pi x)\ dx$$

$$= \int_{0}^{1} x\cos(n\pi x)\ dx + \int_{0}^{1} (-x)\cos(-n\pi x)\ dx$$

$$= \int_{0}^{1} x\cos(n\pi x)\ dx - \int_{0}^{1} x\cos(n\pi x)\ dx = 0$$

Integral 2

$$\int_{-1}^{1} x\sin(n\pi x)\ dx = \left[-\frac{x\cos(n\pi x)}{n\pi} - \int \frac{\cos(n\pi x)}{n\pi}\,dx\right]_{-1}^{1}$$

$$= \left[-\frac{x\cos(n\pi x)}{n\pi} - \frac{\sin(n\pi x)}{n^2\pi^2}\right]_{-1}^{1} = \frac{2}{n\pi}(-1)^{n+1}, \qquad n \in \mathbb{Z}^+$$

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Integral 1

$$2\int_{0}^{1} x^2 = 2\left[\frac{x^3}{3}\right]_0^1 = \frac{2}{3}$$

$$2\int_{0}^{1} x^2\cos(n\pi x)\,dx = 2\left(\left[\frac{x^2\sin(n\pi x)}{n\pi}\right]_0^1 - \frac{2}{n\pi}\int_0^1 x\sin(n\pi x)\,dx\right)$$

$$= -\frac{4}{n\pi}\left(\left[\frac{-x\cos(n\pi x)}{n\pi}\right]_0^1 + \frac{2}{n\pi}\int_0^1 \cos(n\pi x)\,dx\right)$$

$$= \frac{4(-1)^n}{n^2\pi^2}, \qquad n \in \mathbb{Z}^+$$

Integral 2

$$2\int_0^1 x^2 \sin(n\pi x)\,dx = 2\left(\left[-\frac{x^2\cos(n\pi x)}{n\pi}\right]_{-1}^{1} + 2\int_0^1 \frac{x\cos(n\pi x)}{n\pi}dx\right)$$

$$= \frac{2(-1)^{n+1}}{n\pi} + \frac{4}{n\pi}\left(\left[\frac{x\sin(n\pi x)}{n\pi}\right]_{-1}^{1} - \int_0^1 \frac{\cos(n\pi x)}{n\pi}dx\right)$$

$$= \frac{2(-1)^{n+1}}{n\pi} + \frac{4((-1)^n - 1)}{n^3\pi^3}, \qquad n \in \mathbb{Z}^+$$

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Integral 1

Above we found that:

$$2\int_0^1 x^2 \cos(n\pi x)\,dx = \frac{4(-1)^n}{n^2\pi^2}, \qquad n \in \mathbb{Z}^+$$

Thus:

$$2\int_0^1 x^2 \cos(2n\pi x)\,dx = \frac{(-1)^n}{n^2\pi^2}, \qquad n \in \mathbb{Z}^+$$

Integral 2

Above we found that:

$$2\int_0^1 x^2 \sin(n\pi x)\,dx = \frac{2(-1)^{n+1}}{n\pi} + \frac{4((-1)^n - 1)}{n^3\pi^3}, \qquad n \in \mathbb{Z}^+$$

Thus:

$$2\int_0^1 x^2 \sin(2n\pi x)\,dx = \frac{(-1)^{2n+1}}{n\pi} + \frac{4(1-1)}{n^3\pi^3}$$

$$= \frac{-1}{n\pi}, \qquad n \in \mathbb{Z}^+$$

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$$-2\int_0^1 \cos(n\pi x)\,x\,dx = -2\left(\left[\frac{x\sin(n\pi x)}{n\pi}\right]_0^1 - \int_0^1 \frac{\sin(n\pi x)}{n\pi}\,dx\right)$$

$$= 2\left[\frac{\cos(n\pi x)}{n^2\pi^2}\right]_0^1 = \begin{cases} \frac{4}{\pi^2 n^2}, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

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