The Singularity Theorems in General Relativity

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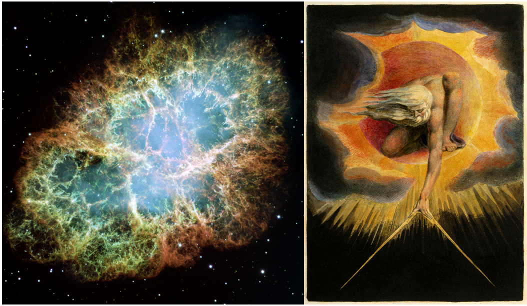
ASC Presentation

Introduction

- General Relativity shows that gravity occurs because of the geometry of spacetime

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu}$$

- Predicts singularities - points where matter density becomes infinite and spacetime ceases to exist
  - Stellar Collapse
  - Beginning of the Universe



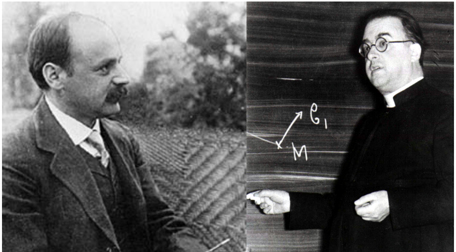
Introduction

Stellar Collapse

- Schwarzschild in 1916 derives the spherically symmetric solution, which has a singularity

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

- Lemaitre (1932) and Synge (1950) realise that the $r = 2M$ surface is not a singularity
- At $r = 0$ we have a true, curvature singularity
- For an observer with $r < 2M$, the singularity will be reached in finite time



Introduction

Stellar Collapse

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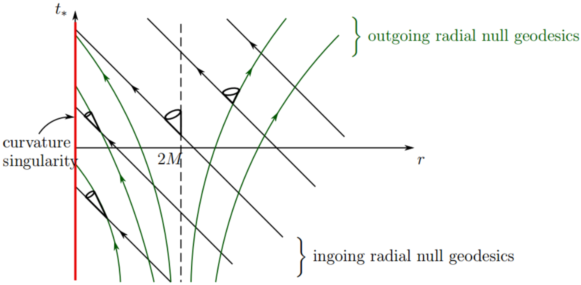
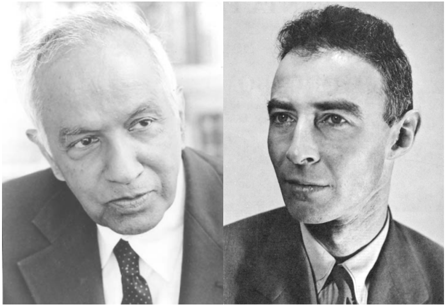


Source: http://www.damtp.cam.ac.uk/user/hsr1000/black_holes_lectures_2014.pdf

Introduction

Stellar Collapse

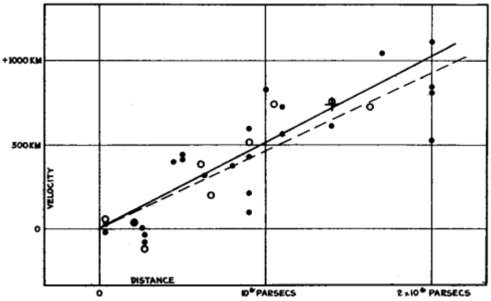
- Chandrasekhar (1931), and Oppenheimer and Volkoff (1939) studied stellar structure, white dwarfs and neutron stars
- Found that stellar collapse was inevitable for sufficiently heavy objects
- Oppenheimer and Snyder (1939) found that a ball of dust will collapse into a Schwarzschild black hole



Introduction

The Beginning of the Universe

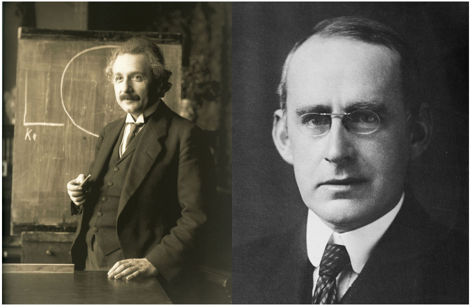
- Friedmann (1924) and Lemaitre (1927) study isotropic and homogenous universes
- Lemaitre (1932) demonstrates general existence of singularities in Bianchi Type I class
- Since Hubble showed the universe was expanding (1929); these models then suggested that the universe had a beginning of infinite density



Introduction

What did these singularities mean?

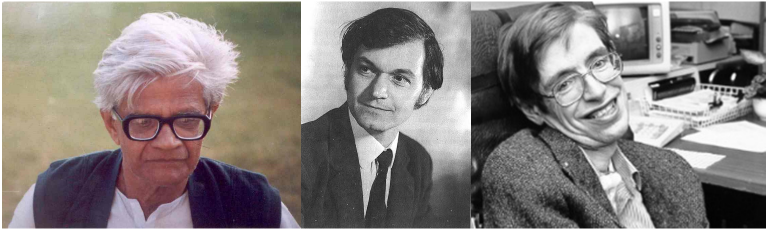
- Early relativists, including Einstein, blamed symmetry assumptions and idealizations about the matter content
- Eddington infamously opposed Chandrasekhar's work
- Singularities were rarely studied and therefore poorly understood
- General study of singularities would have to wait for another generation of relativists



Introduction

What did these singularities mean?

- Raychaudhuri (1955) finds first singularity theorem and developes the Raychaudhuri equation
- Penrose (1965) developes theorem for stellar collapse
- Extended to the early universe by Hawking (1967)
- These results were an integral part of the golden age of general relativity



Singularities

Definition of Singularities

- It is quite difficult to define singularities in GR:
  - Singularities are not part of spacetime
  - Coordinate system choices makes characterization hard
  - Spaces can have singular behaviour whilst the curvature remains bounded
- Because of this, there are many different definitions of singularities

Singularities

Definition of Singularities

- We need a definition using regular points in spacetime
- If a traveller approached a singularity, he would disappear in finite time
- If a curve cannot continue backward in time, matter on the curve must have appeared *ab initio*
- So we say that a singularity in a Lorentzian manifold is a non-spacelike curve which cannot be extended in a regular manner yet only takes part of its cannonical parameter
- This is called (non-spacelike) *b-incompleteness*

Singularities

Structure of the Theorems

Singularity theorems all have a basic form:

If a spacetime $\mathcal{M}$ is sufficiently differentiable, then if

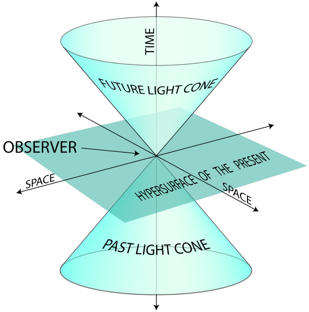
- Condition on curvature
- Causality condition
- Initial/boundary conditions

are satisfied, the spacetime has incomplete causal geodesics

Causality

Causal Structure

- At every point on the manifold we have lightcone



Source: https://upload.wikimedia.org/wikipedia/commons/1/16/World_line.svg

- Gives us timelike, spacelike, and null vectors

Causality

Causal Structure

- $x \ll y$ means there is a timelike curve from x to y
- There is a hierachy of causal conditions
    - Chronology condition forbids closed timelike curves $(x \not\ll x)$
    - Strong causality means that for every $p \in \mathcal{M}$ there is a neighbourhood of p for which no timelike curve passes through more than once
    - Stably causal means that under a small change of the metric, the space will still not violate the chronology condition

Causality

Cauchy Surfaces

- A Cauchy surface $\mathcal{S}$ for a set $\mathcal{N}$ is a surface which every non-spacelike curve in $\mathcal{N}$ intersects exactly once. We say $\mathcal{N}$ is globally hyperbolic
- Theorem: If $\mathcal{S}$ is a Cauchy surface of $\mathcal{N}$, then $\mathcal{N}$ is homeomorphic to $\mathcal{S} \times \mathbb{R}$
- Theorem: Between any two points p, q in $\mathcal{N}$ with $p \ll q$ there is a non-spacelike geodesic of maximum length

Geodesic Focussing

Geodesic Congruences

- Take a spacelike hypersurface $\mathcal{S}$ on our manifold $\mathcal{M}$
- At every point $s \in \mathcal{S}$ there is a timelike unit vector u^α, and so a geodesic $\gamma_p(\tau)$ with $\gamma_p(0) = p$ and $\dot{\gamma}_p(0) = u^\alpha$
- We have a vector field $\dot{\gamma}_p(\tau)$ in some region of spacetime, along with a series of surfaces $\mathcal{S}_\tau$
- The expansion scalar

  $$\theta = \nabla^a \gamma_a(\tau),$$

  where a is the derivative on S_τ, is the infinitessimal expansion of S_τ
- If $\theta > 0$ the surface is expanding, if $\theta < 0$ the surface is shrinking

Geodesic Focussing

Geodesic Congruences

- From the Raychaudhuri equation, we can derive

$$\frac{d\theta}{d\tau} \leq -\frac{1}{3}\theta^2 - R_{\alpha\beta}u^\alpha u^\beta$$

- If $R_{\alpha\beta}u^\alpha u^\beta \geq 0$, then

$$\frac{d\theta}{d\tau} \leq -\frac{1}{3}\theta^2 \leq 0$$

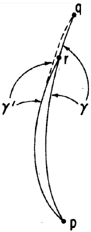
  so gravity is attractive!

- If $\theta < 0$ at some τ_0, then θ will reach negative infinity at finite affine parameter

Geodesic Focussing

Geodesic Congruences

Theorem: A timelike geodesic curve between two points cannot be maximal if there is such a focal point



Geodesic Focussing

Energy Conditions

- If $R_{\alpha\beta}u^\alpha u^\beta \geq 0$, then by the Einstein field equation

$$T_{\alpha\beta}u^\alpha u^\beta \geq \frac{1}{2}T$$

- This is known as the strong energy condition
- Seems to hold in most physically relevant scenarios
- For a perfect fluid with density of water, pressure would need to be less than -10^{15} atmospheres to violate the condition

Cosmological Theorems

The story so far

Geodesic Focussing

- If strong energy condition holds, converging geodesics will focus at finite parameter
- If a geodesic has a focus point between two points p, q, then there is a longer path from p to q

Causality

- Between two points p, q in a globally hyperbolic set $\mathcal{N}$, there is a geodesic of maximum length.

Cosmological Theorems

First Theorem

Theorem

Let $\mathcal{M}$ be a globally hyperbolic manifold satisfying the strong energy condition. Suppose $\mathcal{S}$ is a Cauchy surface for which the geodesic expansion $\theta \leq C \leq 0$ everywhere. Then no geodesic passing through $\mathcal{S}$ can be extended further into the past then $3/|C|$.

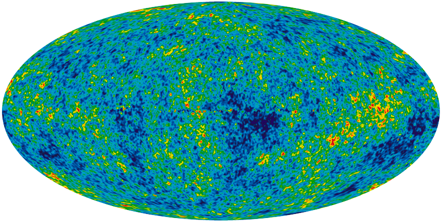
Proof Sketch

Assume γ is a geodesic which can be extended further than $3/|C|$ into the past, and let p be a point on this geodesic lying further than $3/|C|$ in the past. Then there is a maximum length path from p to $\mathcal{S}$ with length above $3/|C|$. But Raychaudhuri equations shows any curve of this length must contain a focal point.

Cosmological Theorems

Initial Conditions for Cosmology

- Cosmological observations show that the universe is expanding
- In the ΛCMD model, the universe was fairly uniformally expanding at the time of nucleosynthesis, $z \approx 10^8$.
- Our theorem suggests that either
  - The universe had a singular beginning
  - The strong energy condition is incorrect
  - The universe is not globally hyperbolic



Cosmological Theorems

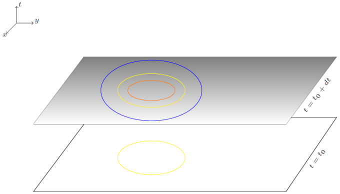
Generalizations

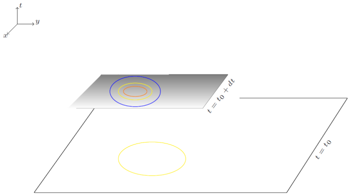
- More sophisticated versions exist
- Hawking (1967) replaced globally hyperbolic with strongly causal and removed the expanding everywhere assumption
- Hawking (1967) removed the globally hyperbolic conditions, assumes existence of compact spacelike three-surface. Weakens conclusion to the existence of an incomplete geodesic.
- Weakening causality condition cannot save us?

Stellar Collapse

Trapped Surfaces

- A closed trapped surface is intuitively a surface so strongly bound that light cannot escape
- Because they occur in the Schwarzschild solution, they will occur in any stellar collapse which is sufficiently spherical





Source: http://arxiv.org/abs/1410.5226

Stellar Collapse

The Theorems

Theorem (Penrose 1965):

If spacetime has a non-compact Cauchy surface and a closed future-trapped surface, and if the strong energy condition holds, then there are future incomplete null geodesics

Theorem (Hawking and Penrose 1970):

If the strong, chronology and generic conditions hold and if there is a closed trapped surface, then spacetime is causal geodesically incomplete

- The generic condition requires that

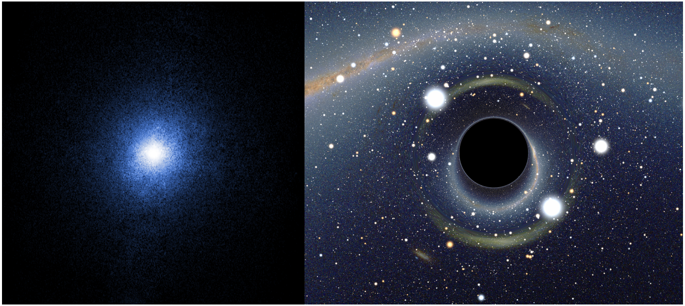
$$K_{[a}R_{b]cd[e}K_{f]}K^cK^d \neq 0$$

for some tangent vector K^a to every geodesic.

Stellar Collapse

Black Holes

- What happens to these stars?
- They will settle down to become black holes
  - Schwarzschild solution
  - Kerr solution



Conclusion

What next?

- We seem to have shown there are singularities
    - Stellar collapse
    - The beginning of the universe
- What is the nature of these singularities?

Conclusion

What next?

Possibility 1

The singularity is a region of infinite curvature and density.

Possibility 2

A more subtle pathology occurs, perhaps causal conditions are violated.

Possibility 3

General relativity and/or the strong energy condition breaks down - a more complete theory (quantum gravity?) is needed.

Conclusion

What next?

- Singularity theorems are still active
- Can we apply the theorems to different physical situations?
- Can we relax assumptions in the theorems?
- Can we better characterize singularities?

Conclusion

What next?

- Thank you for listening
- Any questions?