ASC Report: Advanced General Relativity

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ASC Report Structure

Since my ASC this semester has been a reading course rather than a research course, this 'ASC Report' is just a collation of the assessment that I have been given throughout the semester. There are three tasks which extend upon the readings done from *Techniques of Differential Topology in Relativity* by Penrose, and *The Large Scale Structure of Space-Time* by Hawking and Ellis. After this is a textbook review, of *Quantum Field Theory* by Padmanabhan. Lastly, there are a collection of exercises taken from *An Introduction to General Relativity* by Hughston and Tod.

Contents

Task 1

Definition of Strong Causality

Both *Techniques of Differential Topology in Relativity* by R. Penrose, and *The Large Scale Structure of Space-Time* by Hawking and Ellis, a notion of strong causality is defined. The definitions given are quite different, so we shall show that in spite of this, they are equivalent.

According to Penrose, an open set Q is causally convex if it intersects no trip on a disconnected set. As noted in the footnote to the definition, this is equivalent to requiring that if $x, z \in Q$ and $x \ll y \ll z$, then $y \in Q$. A manifold (M, g) is strongly causal if at every point $p \in M$ there are arbitrarily small causally convex neighbourhoods of p.

According to Hawking and Ellis, M is strongly causal if every neighbourhood of p contains a neighbourhood of p for which no non-spacelike curve intersects more than once. We shall show that these definitions are equivalent.

We begin by showing that the Penrose definition implies the Hawking and Ellis definition. Take a point $p \in M$ and any open neighbourhood U of p. If M is strongly causal according to the Penrose definition, there is an open causally convex set V containing p so that $V \subset U$. Choosing a simple region (an open simply convex set contained in a compact set) N of p, we have that $N \cap V$ is an open non-empty subset of V, since it contains p. Now choose $x, y \in N \cap V$ so that $x \ll p \ll y$; that this is always possible is proven in proposition 4.9 of Penrose. Then $\langle x, y \rangle$ is a causally convex subset of V, and is open by proposition 4.7 of Penrose.

Let $\gamma : [0, 1] \to M$ be a non-spacelike curve so that $\gamma(a), \gamma(b) \in \langle x, y \rangle$. Then using propositions 2.20 and 2.23 in Penrose, if $t_0 < t_1$, then $\gamma(t_0) \prec \gamma(t_1)$. So for any $t \in [a, b]$ we have the relationships

$$x \ll \gamma(a) \prec \gamma(t) \prec \gamma(b) \ll y.$$

Now apply proposition 2.18. which states that $a \ll b \prec c$ implies $a \ll c$ and likewise for $a \prec b \ll c$. This means that

$$x \ll \gamma(t) \ll y.$$

Then, by the definition of the set $\langle x, y \rangle$, we have that $\gamma(t) \in \langle x, y \rangle$ for every $t \in [a, b]$, so γ can only intersect $\langle x, y \rangle$ once; ie, on a single interval in $[0, 1]$. We have proven that every open neighbourhood U of p contains a neighbourhood $\langle x, y \rangle$ of p which every non-spacelike curve intersects at most once. Hence the Penrose definition implies the Hawking and Ellis definition of strong causality.

Now we prove the converse. Let M be strongly causal in the Hawking and Ellis sense, and take $p \in M$. For any open neighbourhood U of p, there is a open subset V containing p so that every non-spacelike curve intersects V at most once. As every trip is a non-spacelike curve, this means that V is causally convex and hence that the M is strongly converse under the Penrose definition.

Task 2

Singularities in the FRW Metrics

2.1 Introduction

The Friedmann-Robertson-Walker spaces are those spaces which are spaticially homogeneous and spherically symmetric around every point. Coordinates can be chosen so that the metric has form

$$ds^2 = -dt^2 + S^2(t)d\sigma^2$$

where $d\sigma^2$ is a time-independent metric of a 3-space of constant curvature, and where $S(t) > 0$. This metric becomes singular if $S(t) = 0$ at any time t. It also is singular if $S(t)$ or its first or second order derivatives become infinite at any finite time. The former singularity can be thought of as the 'big bang' or 'big crunch' since the universe will at this time be contracted down infinitely; later we shall see that this means that the matter density will become infinite. When S instead becomes infinite, we have a 'big rip' scenario where the universe expands infinitely; we shall also see that in this scenario, the matter density becomes 0.

By rescaling $S(t)$, the geometry of the 3-space can be chosen to have curvature K of 0 or ± 1. We can then write

$$d\sigma^2 = d\chi^2 + f^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)$$

with

$$f(\chi) = \begin{cases} \sin\chi & \text{if } K = 1 \\ \chi & \text{if } K = 0 \\ \sinh\chi & \text{if } K = -1 \end{cases}$$

2.2 Energy Conditions

Given K, the function $S(t)$ and cosmological constant Λ, the form of the energy-momentum tensor is completely determined by the Einstein field equations. Because of the high degree of symmetry of the FRW spaces, the energy-momentum tensor is a perfect fluid with density $\rho(t)$ and pressure $p(t)$. Conservation of energy can be expressed as

$$\dot{\rho} = -3(\rho + p)\frac{\dot{S}}{S}. \tag{2.1}$$

We will assume the weak, dominant and strong energy conditions. For perfect fluids these are the inequalities

$$\begin{array}{ll} \text{Weak:} & \rho \geq 0, \quad \rho + p \geq 0 \\ \text{Dominant:} & \rho \geq |p| \\ \text{Strong:} & \rho + p \geq 0, \quad \rho + 3p \geq 0 \end{array} \tag{2.2}$$

Let us assume that the universe is expanding, so that $\dot{S} \geq 0$. Because of the time reversal invariance of the equations, this is without loss of generality, as the result we derive will also hold when time-reversed for a contracting universe. From the dominant energy condition we can derive the inequality

$$p + \rho \leq 2\rho$$

which when applied to (2.1) gives us

$$-6\rho\frac{\dot{S}}{S} \leq \dot{\rho} \leq 0.$$

The right-hand inequality means that the energy density must decrease as the expansion factor increases. The left-hand inequality can be rearranged to

$$-6\frac{\dot{S}}{S} \leq \frac{\dot{\rho}}{\rho}$$

which upon integration leads to

$$6\ln(S(t_1)) - 6\ln(S(t_0)) \leq \ln(\rho(t_0)) - \ln(\rho(t_1))$$

for $t_1 \geq t_0$. We can then exponentiate to find

$$\left(\frac{S(t_1)}{S(t_2)}\right)^6 \leq \frac{\rho(t_1)}{\rho(t_2)}.$$

We therefore have the inequalities

$$\rho(t) < \frac{\rho(t_0)S(t_0)^6}{S(t)^6} \text{ for } t < t_0, \qquad \rho(t) > \frac{\rho(t_0)S(t_0)^6}{S(t)^6} \text{ for } t > t_0 \tag{2.3}$$

which hold if the universe is expanding between the two times considered. Crucially, in an expanding universe if $\rho(t_0) = 0$ at some time t_0 then the first inequalitiy implies ρ must be zero at all earlier times. Since the energy is positive and monotonically decreases, this would require $\rho = 0$ at all times. Time-reversal symmetry means that this result also holds for contracting universes, so we conclude that if the universe has zero energy density at any time, then the energy density must always be zero. This is of course a special case of the conservation theorem on pg 94 of Hawking and Ellis.

Using the strong energy condition we can derive the inequality

$$\frac{2}{3}\rho \leq p + \rho.$$

When applied to (2.1) we get

$$\dot{\rho} \leq -2\rho\frac{\dot{S}}{S}$$

and so

$$\frac{\dot{\rho}}{\rho} \leq -2\frac{\dot{S}}{S}.$$

This can be integrated to

$$\ln(\rho(t_1)) - \ln(\rho(t_0)) < 2\ln(S(t_0)) - 2\ln(S(t_1)),$$

for $t_1 > t_0$. So upon exponentiation of both sides, we get the inequalities

$$\rho(t) > \frac{\rho(S(t_0))S(t_0)^2}{S(t)^2} \text{ for } t < t_0, \qquad \rho(t) < \frac{\rho(S(t_0))S(t_0)^2}{S(t)^2} \text{ for } t > t_0. \tag{2.4}$$

The above inequalities require both the energy density to become infinite as $S \to 0$, and 0 as $S \to \infty$. Therefore these singularities of the FRW metric correspond to singular behaviour of the matter density.

2.3 Qualitative Behaviour of the Friedmann equations

The Friedmann equations express the relationships between S, ρ and p:

$$\frac{\dot{S}^2 + K}{S^2} = \frac{8\pi\rho + \Lambda}{3} \tag{2.5}$$

$$\frac{\ddot{S}}{S} = -\frac{4\pi}{3}(\rho + 3p) + \frac{\Lambda}{3}. \tag{2.6}$$

These are not independent, but rather the second equation can be derived from combining the first equation with (2.1). We shall examine the evolution of these equations for matter satisfying all of the energy conditions. In particular, we are interested in singularities. Using the second equation, it is apparent that an infinite value of $\dot{S}$ or $\ddot{S}$ would require an infinite value of S. So we only need worry about the $S = 0$ and $S \to \infty$ singularities. We will split our analysis of the FRW universe into a couple of parts.

2.3.1 Extrema and Stationary Solutions

An extrema of S occurs if $\dot{S}(t) = 0$ for some t. We shall analyse under which conditions extrema are possible. Applying (2.5), we have

$$\frac{K}{S(t)^2} = \frac{8\pi\rho + \Lambda}{3}$$

at an extrema, so applying the positivity of energy density, we find that

$$\frac{K}{S(t)^2} - \frac{\Lambda}{3} \geq 0$$

at an extrema, with equality only if $\rho = 0$. If $\Lambda \geq 0$, this inequality can only be satisfied if $K = 1$, or if $\Lambda = K = \rho(S(t)) = 0$. Applying (2.3), this last possibility requires $\rho = 0$ at all times, and so this solution corresponds to Minkowski space-time, a stationary FRW universe.

A stationary FRW universe has the additional requirement that $\ddot{S} = 0$, and so using (2.6),

$$\frac{\Lambda}{3} = \frac{4\pi}{3}(\rho + 3p).$$

Since the right-hand side must be non-negative, the stationary FRW universes must have $\Lambda \geq 0$. The stationary FRW universes are summarised in Table 1.

Universe Name	Curvature	Cosmological Constant	Matter Content
Minkowski	$K = 0$	$\Lambda = 0$	$\rho = p = 0$
Spherical	$K = 1$	$\Lambda \geq 0$	$\Lambda = 4\pi(\rho + 3p)$

Table 1: Stationary FRW Universes

2.3.2 Past of an Expanding Universe

Let us assume that at some t_1, the universe is expanding and so $\dot{S}(t_1) > 0$. First considering the case of $\Lambda \leq 0$, the third inequality of (2.2) applied to (2.6) gives us the inequality

$$\ddot{S}(t) \leq 0.$$

Therefore $\dot{S}(t_0) \geq \dot{S}(t_1)$ if $t_0 \leq t_1$, and so

$$S(t_0) = S(t_1) - \int_{t_0}^{t_1} \dot{S}(t)dt \leq S(t_1) - (t_1 - t_0)\dot{S}(t_1).$$

Taking t_0 to be sufficiently small, we can force $S(t_0) < 0$, and so a singularity inevitably occured in the past.

Now take the case of $\Lambda > 0$. We have three possible behaviours of $S(t)$ in the past.

1. There is an $S_0 < S(t_1)$ at which an extrema occurs.
2. As $t \to -\infty$, $S(t)$ assymptotes to but never reaches 0.
3. There is a time $t_0 < t_1$ so that $S(t_0) = 0$ and therefore we have an initial singularity.

We have already shown that the only universes with extrema for $\Lambda > 0$ have $K = 1$.

In the second and third scenario, we can rearrange (2.5) to

$$\frac{\dot{S}dt}{\sqrt{\frac{8\pi}{3}\rho S^2 + \frac{\Lambda}{3}S^2 - K}} = dt.$$

This allows us to calculate the time since the universes origin as

$$\int_0^{S(t_1)} \frac{\dot{S}dt}{\sqrt{\frac{8\pi}{3}\rho S^2 + \frac{\Lambda}{3}S^2 - K}} = t_1 - t_0.$$

If the left-hand side integral does not converge, then we are in the second scenario, otherwise we are in the third. The only possible way for the left-hand side integral to not converge is if

$$\frac{1}{\sqrt{\frac{8\pi}{3}\rho S^2 + \frac{\Lambda}{3}S^2 - K}} > \frac{M}{S}$$

for some constant M. This can be rearranged to give

$$S^2 > M^2\left(\frac{8\pi}{3}\rho S^2 + \frac{\Lambda}{3}S^2 - K\right) > M^2\left(\frac{8\pi}{3}\rho S^2 - K\right)$$

since $\Lambda > 0$. Making ρ the subject of the inequality, we get

$$\rho(t) < \frac{3}{8\pi M^2} + \frac{3K}{8\pi S(t)^2}.$$

First note that if $K = -1$, this inequality cannnot be satisfied for a positive energy density, so any expanding universe with $K = -1$ has an initial singularity. If instead $K = 0$, using (2.4) allows us to deduce that this inequality cannot be satisfied, since the inequality would prevent $\rho \to \infty$ as $S \to 0$. Finally, if $K = 1$, using (2.4) we get

$$\frac{A}{S(t)^2} < \rho(t) < \frac{3}{8\pi M^2} + \frac{3K}{8\pi S(t)^2}.$$

for some A, which means that while it is possible to assymptote to 0 if $K = 1$, this significantly constrains the form of $\rho(t)$.

So we have found that if a universe satisfies both the strong and weak energy conditions, and if the universe is expanding at some time, then for the universe to be singularity free it is necessary for $K = 1$ and $\Lambda > 0$. All our conclusions also hold under time reversal, so for a universe that is contracting, then the only possibility to avoid a final singularity is if $K = 1$ and $\Lambda > 0$.

2.3.3 Evolution of FRW Universe near a Singularity

We now turn to the problem of describing the initial expansion of a universe. The first Friedmann equation can be rearranged to

$$\dot{S} = \sqrt{\frac{8\pi}{3}\rho(t)S^2 + \frac{\Lambda}{3}S^2 - K} \underset{S\to 0}{\rightarrow} S\sqrt{\frac{8\pi}{3}\rho(t)}.$$

since using (2.4),

$$\frac{A}{S(t)^2} < \rho(t)$$

and so $S^2\rho$ is bound away from zero as $S \to 0$. We now apply (2.3) and (2.4) to get the inequality

$$\frac{A^3}{3S^2} \geq \dot{S} \geq B$$

for constants A and B. Integrating these inequalities from $t = 0$, $S(0) = 0$ gives us

$$At^{1/3} \geq S(t) \geq Bt.$$

2.3.4 Future of an Expanding Universe

Again we assume that $\dot{S}(t_1) > 0$ for some time t_1, but now we shall explore the future of the universe. We shall begin by ruling out the $S \to \infty$ by showing that the assymptotic growth of the universe is at most exponential. Using (2.5),

$$\dot{S} = \sqrt{\frac{8\pi}{3}\rho S^2 + \frac{\Lambda}{3}S^2 - K} \leq S\sqrt{\frac{1}{3}(8\pi\rho + \Lambda) - \frac{K}{S^2}} \leq S\sqrt{\frac{1}{3}(8\pi\rho + \Lambda) + \frac{1}{S^2}}.$$

Since ρ decreases with increasing S, the expression under the square-root sign decreases with increasing S, and so for any $S > S(t_1)$ will be bounded by some C_1. We therefore have that

$$\dot{S} \leq C_1 S \text{ for } S \geq S(t_1)$$

and so

$$S(t) \leq e^{(t-t_1)C_1} S(t_1) \text{ for } t \geq t_1.$$

This rules out the big rip scenario.

Now let us explore other possible future behaviour. First take the case of $\Lambda \geq 0$ and $K \neq 1$. We have already showed that no extrema of S can occur in this situation, so the universe must continue to expand, and become arbitrarily large as time becomes infinite.

More specifically, if $\Lambda > 0$, (2.5) gives us

$$\dot{S} = \sqrt{\frac{8\pi}{3}\rho S^2 + \frac{\Lambda}{3}S^2 - K} \geq S\sqrt{\frac{8\pi}{3}\rho + \frac{\Lambda}{3}}.$$

Since (2.4) implies that $S^2\rho \to 0$ as $S \to \infty$, for sufficiently large S we have

$$\dot{S} \approx S\sqrt{\frac{\Lambda}{3}}$$

and so the universe will grow exponentially as $S(t) \propto e^{t\sqrt{\frac{\Lambda}{3}}}$.

If instead $\Lambda = 0$, we find

$$\dot{S} = \sqrt{\frac{8\pi}{3}\rho S^2 - K}$$

so applying (2.3) and (2.4), we find that for $K = -1$,

$$S(t) \propto t.$$

For the case $K = 0$, we get

$$\dot{S} = \sqrt{\frac{8\pi}{3}\rho S^2}$$

which is the same equation we dealt with in the previous section, except now S is large rather than small. So we can use the reverse inequalities of (2.3) and (2.4) to the ones used in that section to get

$$\frac{A^3}{3S^2} \leq \dot{S} \leq B$$

for constants A and B, and therefore by integration find that

$$At^{1/3} \leq S(t) \leq Bt$$

for sufficiently large t.

Now we shall deal with the case where $\Lambda < 0$, so that (2.6) combined with (2.2) gives us

$$\ddot{S} < \frac{\Lambda}{3}S.$$

So if the universe is larger at t_2 then t_1, we find

$$\dot{S}(t_2) < \dot{S}(t_1) + (t_2 - t_1)\frac{\Lambda}{3}S(t_1).$$

Making t_2 sufficiently large results in $\dot{S}(t_2) < 0$, at which point the universe is contracting. As a result, the universe for $\Lambda < 0$ has both initial and final singularities.

We are now left with the case of $K = 1$, $\Lambda \geq 0$. First let $\Lambda = 0$, so that (2.5) gives us

$$\dot{S}^2 = \frac{8\pi}{3}S^2\rho - 1.$$

Applying (2.4), $S^2\rho$ monotoncilly decreases, so if for sufficiently large S

$$S^2\rho \leq \frac{3}{8\pi}$$

then there will be an extrema of S. At this extrema, the energy conditions require $\ddot{S} \leq 0$, so if $\ddot{S} = 0$ then universe will assymptote to a stationary universe, and otherwise the universe will recollapse. If instead

$$S^2\rho > \frac{3}{8\pi}$$

for every S, then the universe will grow without bound, at most linearly in t.

Finally take the case of $K = 1$, $\Lambda > 0$. This change makes it harder for an extrema to occur as we now have

$$\dot{S}^2 = \frac{8\pi}{3}S^2\rho + \frac{S^2\Lambda}{3} - 1$$

which has an additional positive term to the previous case. A further complication is added by the fact that $\ddot{S}$ may be positive at the extrema, so that a extrema can exist and yet still the universe grows without bound. Otherwise, the same analysis applies; the universe could grow without bound, assymptote to a stationary universe, or recollapse. So the evolution of a universe with $K = 1$, $\Lambda > 0$ can in general be quite complicated; the only certainty is that for sufficiently large Λ, the universe will grow exponentially.

2.3.5 Summary

Table 2 summarises the types of universes we have found. The only singularities that occur are those where $S \to 0$. However, singularities of this kind occur in the past of all of the universes considered except for the stationary cases and the case of $K = 1$, $\Lambda > 0$ universe. The future behavior of expanding universes is more varied, though can still be predicted without knowledge of the matter content, except when $K = 1$ and $\Lambda \geq 0$.

Λ	K	Matter Content	Initial Singularity	Future Behaviour
< 0	All	Any	Yes	Singularity
$= 0$	-1	Any	Yes	$\propto t$
$= 0$	0	Any	Yes	$\propto t^\gamma,\ \gamma \in [\frac{1}{3}, 1]$
$= 0$	1	$\forall t,\ 8\pi S^2\rho > 3$	Yes	Linearly Bounded Growth
$= 0$	1	Stationary Extrema	Yes	Asymptotic to Spherical Universe
$= 0$	1	Extrema	Yes	Singularity
> 0	$-1, 0$	Any	Yes	$\propto e^{t\sqrt{\frac{\Lambda}{3}}}$
> 0	1	$\forall t,\ 8\pi S^2(\rho + \Lambda) > 3$	Yes	Varied, for large Λ, $\propto e^{t\sqrt{\frac{\Lambda}{3}}}$
> 0	1	Extrema	Maybe	Varied, for large Λ, $\propto e^{t\sqrt{\frac{\Lambda}{3}}}$
$= 0$	0	$\rho = p = 0$	No	Stationary
≥ 0	1	$\Lambda = 4\pi(\rho + p)$	No	Stationary

Table 2: A summary of all FRW universes satisfying the energy conditions and which either are expanding or stationary. Contracting FRW universes can be found by time reversing the expanding universes.

Task 3

Two-Dimensional Misner Space

3.1 Introduction

We shall study the two-dimensional Misner space-time and its extensions. Misner space $\mathcal{M}$ is given by the metric

$$ds^2 = -\frac{dt^2}{t} + tdx^2 \tag{3.1}$$

for $t > 0$ and $x \in [0, 2\pi]$. We identify $x = 0$ and $x = 2\pi$ so that the topology of the space is $\mathbb{R} \times S^1$.

This metric becomes singular at $t = 0$ and so we shall analyse this singularity by examining the geodesics of the space. Before doing so it will be useful to mention two symmetries of the space. The metric is invariant under translation of x by an arbitrary constant, $x \to x + x_0$, since the metric does not depend on x. Less trivially, there is also a scale symmetry,

$$t \to \lambda t, \quad ds^2 \to \lambda ds^2.$$

3.2 Geodesics of Misner Space

A particle moving along a geodesic minimises the action given by

$$A = \int_{\tau_0}^{\tau_1} -\frac{\dot{t}^2}{t} + t\dot{x}^2.$$

Using the translation invariance of the metric, we find that the quantity

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = 2t\dot{x}$$

is constant, and so there is a constant P such that

$$\dot{x} = \frac{P}{t}. \tag{3.2}$$

Null geodesics satisfy

$$-\frac{\dot{t}^2}{t} + t\dot{x}^2 = 0$$

which gives us the equation

$$\dot{t}^2 = t^2\dot{x}^2.$$

Combining this with (3.2) gives us

$$\dot{t}^2 = P^2$$

so $t = \pm P\tau + C$. We shall assume that $t(0) = 1$, so that $C = 1$. Furthermore, we can set $P = 1$ since this simply scales the affine parameter. We can now integrate

$$\dot{x} = \pm\frac{1}{t} = \pm\frac{1}{\tau + 1}$$

to find that the path of the geodesics are given by

$$t = \tau + 1, \quad x = \pm \log{(\tau + 1)} + K. \tag{3.3}$$

These solutions correspond to left and right spirallying geodesics. The choice of K just corresponds to the location of the geodesic when $\tau = 0$. From these solutions we find that the distance between the singular point $t = 0$ and the regular point $(1, 0)$ can be traversed in a finite affine parameter. The solutions do however pass through the time-like surface $x = 0$ an infinite number of times before reaching the singularity.

If we instead take time-like curves, we can choose the parameterization so that

$$-\frac{\dot{t}^2}{t} + t\dot{x}^2 = -1$$

and hence

$$\dot{t}^2 = t^2\dot{x}^2 + t.$$

Now applying (3.2), we find that

$$\dot{t} = \pm\sqrt{P^2 + t}.$$

We shall assume that t increases with increasing τ, so that

$$\frac{dt}{\sqrt{P^2 + t}} = d\tau.$$

Upon integrating this expression we find that

$$\tau + C = 2\sqrt{P^2 + t}.$$

We shall demand, without loss of generality, that at proper time $\tau = 0$ the time-coordinate will be $t = 1$, and hence

$$C = 2\sqrt{P^2 + 1}.$$

Rearranging, we can find an expression for the coordinate time in terms of proper time:

$$t = \frac{(\tau + 2\sqrt{P^2 + 1})^2}{4} - P^2 = (\tau - 2P + 2\sqrt{P^2 + 1})(\tau + 2P + 2\sqrt{P^2 + 1}).$$

We can now integrate (3.2) to find that

$$x(\tau) = \int \frac{P}{t}\, d\tau = \int \frac{4P}{(\tau + 2\sqrt{P^2 + 1})^2 - 4P^2}\, d\tau$$

$$= \log(\tau + 2P - 2\sqrt{P^2 + 1}) - \log(\tau - 2P - 2\sqrt{P^2 + 1}) + K.$$

$$= \log\left(\frac{\tau - 2P + 2\sqrt{P^2+1}}{\tau + 2P + 2\sqrt{P^2+1}}\right).$$

So our equations for the trajectory of a time-like geodesic are

$$t = \frac{(\tau + 2\sqrt{P^2+1})^2}{4} - P^2 \quad x = \log\left(\frac{\tau - 2P + 2\sqrt{P^2+1}}{\tau + 2P + 2\sqrt{P^2+1}}\right) \tag{3.4}$$

The constant K simply determines the location of $x(0)$, but since the space is symmetric under spacial translations, we have set the constant to zero without loss of generality.

The value of P is quite crucial to the trajectory of the geodesics. Since at $\tau = 0$,

$$\dot{x}(0) = \frac{P}{t(0)} = P,$$

we can interpret P as the velocity of the observer at $\tau = 0$. The sign of P determines whether the observer is travelling leftward or rightward. If $P > 0$, the observer reaches $t = 0$ when

$$\tau = 2P - 2\sqrt{P^2+1} < 0.$$

As $\tau \to 0^+$, we find that $x(\tau) \to \infty^+$, but since we identify the point x with $x + 2\pi$, this means that as $t = 0$ is approached, the observer spirals more and more quickly to the right. They reach $t = 0$ in finite proper time.

If instead $P < 0$, then the only effect this has that the sign of $\dot{x}$ is reversed, so that $x(\tau)$ now spirals leftward rather than rightward. But for the case of $P = 0$,

$$x(\tau) = \log\left(\frac{\tau+2}{\tau+2}\right) = 0$$

and hence this geodesic reaches $t = 0$ without any motion in the spacial direction.

3.3 Extending the Misner Metric

We can extend our metric by defining $y = x - \log(t)$. The metric then becomes

$$ds^2 = 2dydt + tdy^2$$

which is non-degenerate and analytic for all values of t. Our geodesic equations remain the same when $t > 0$, allowing us to deduce that time-like geodesics are given by equations

$$t(\tau) = \frac{(\tau + 2\sqrt{P^2+1})^2}{4} - P^2 = (\tau - 2P + 2\sqrt{P^2+1})(\tau + 2P + 2\sqrt{P^2+1})$$

$$y(\tau) = x(\tau) - \log(t) = \log\left(\frac{\tau - 2P + 2\sqrt{P^2+1}}{\tau + 2P + 2\sqrt{P^2+1}}\right) - \log\left(\frac{(\tau + 2\sqrt{P^2+1})^2}{4} - P^2\right)$$
$$= -2\log\left(\tau + 2P + 2\sqrt{P^2+1}\right).$$

If $P \leq 0$, then since as $\tau \to \tau + 2P + 2\sqrt{P^2+1}$ from the right then $y(\tau) \to -\infty$, we find that these geodesics reach $t = 0$ at finite proper time, but cannot be extended through the surface. Instead they spiral leftward, passing through the surface $y = 0$ an infinite number of times.

If $P > 0$, then the first root of $t(\tau)$ will occur before this point. Following these geodesics backward in time, they pass through the surface $t = 0$ with no difficulty, but eventual reach the turning point of $t(\tau)$. From there, the time coordinate begins to grow again, bring the geodesic to the surface $t = 0$ from the $t < 0$ side. As the surface is approached, the geodesic spirals leftward at an ever increasing pass, reaching the surface $t = 0$ at finite proper time but having passed throught the surface $y = 0$ an infinite number of times. So our extension of the spacetime violates causality, since we can construct closed time-like loops in the space $t < 0$.

For null geodesics, our equations become

$$\begin{aligned} t(\tau) &= \tau + 1 \\ y(\tau) = x(\tau) - \log(t) &= K \pm \log(\tau + 1) - \log(\tau + 1) \\ &= K \text{ or } K + \log\left(\frac{1}{\tau(\tau + 1)}\right). \end{aligned}$$

So in this coordinate system, the rightward spiralling null geodesic has become fully extendible. The leftward geodesic remains inextendible, executing the same behaviour as the $P < 0$ time-like geodesics.

The behaviour of null geodesics on the surface $t = 0$ is pathological. This is because the surface $t = 0$ is null and compact, so there is a null geodesic trapped in this surface, given by equations

$$t(\tau) = 0, \quad x(\tau) = \tau.$$

The scaling of the affine parameter is of course arbitrary. So even just extending $\mathcal{M}$ to $\overline{\mathcal{M}}$ results in a violation of the causality conditions.

If we can extend Misner 2-space so that the rightward travelling null geodesics become extendable, then it is logical to ask whether we can instead extend the space so that leftward travelling null geodesics are extendable. This can be achieved by making the transformation $z = x + \log(t)$. This changes the metric to

$$ds^2 = -2dydt + tdy^2.$$

Notice that this space can be turned into the other extension via the mapping $z \to -y$. So the behaviour of geodesics in this extension are identical to that of the previous extension, only parity reversed.

3.4 Causality and Misner Space

We have found two inequivalent extensions of our metric, neither of which are able to extend all of the geodesics. Furthermore, both exhibited chronological violations. We will now show that in general, any extension of Misner space must violate a causality condition, namely that it cannot be future distinguishing. This result requires only an extremely weak condition on the extension. We shall furthermore show that closed null loops will occur provided that the boundary of $\mathcal{M}$ in the extension is not too pathological.

Let us take our original metric (3.1) on $\mathcal{M}$, and take the point $(1, 0)$. From (3.4) we know the geodesics are given

$$t = \frac{(\tau + 2\sqrt{P^2 + 1})^2}{4} - P^2, \quad x = \log\left(\frac{\tau - 2P + 2\sqrt{P^2 + 1}}{\tau + 2P + 2\sqrt{P^2 + 1}}\right) - \log\left(\frac{-P + \sqrt{P^2 + 1}}{P + \sqrt{P^2 + 1}}\right).$$

If we then substitute $\tau = 2\sqrt{e^\pi + P^2} - 2\sqrt{P^2 + 1}$, we find that at this value of τ,

$$t(\tau) = e^\pi, \quad x(\tau) = \log\left(\frac{\tau - 2P + 2\sqrt{P^2 + 1}}{\tau + 2P + 2\sqrt{P^2 + 1}}\right) - \log\left(\frac{-P + \sqrt{P^2 + 1}}{P + \sqrt{P^2 + 1}}\right)$$

$$= \log\left(\frac{-P+\sqrt{e^{\pi}+P^2}}{P+\sqrt{e^{\pi}+P^2}}\right) - \log\left(\frac{-P+\sqrt{P^2+1}}{P+\sqrt{P^2+1}}\right).$$

As P varies along $\mathbb{R}$, $x(\tau)$ can take any value on the interval $(-\pi, \pi)$. So we find that the chronological future of $(1, 0)$ contains the set $(e^{\pi}, \infty) \times [0, 2\pi]$. Applying the conformal and translation symmetry of Misner space, in general $I^+(t_1, x_1)$ contains all of $\mathcal{M}$ where $t > t_1 e^{\pi}$.

Now imagine that we can extend $\mathcal{M}$ into some bigger space $\mathcal{M}'$. Since the future of $\mathcal{M}$ is geodesically complete, $\mathcal{M}$ is a future set and so by Proposition 2.1 in Penrose,

$$I^+(\overline{\mathcal{M}}) = I^+(\mathcal{M}) = \mathcal{M}.$$

Let us furthermore assume that there is a time-like path γ from some $x \in \partial M$ to some $y \in \mathcal{M}$. Then for every $\tau \geq 0$,

$$I^+(\gamma(\tau)) = I^+(t_\gamma(\tau), x_\gamma(\tau)) \subset I^+(\gamma(0)).$$

Since as $\tau \to 0$ we know that $t_\gamma(\tau) \to 0$, for any point $(t_1, x_1) \in \mathcal{M}$ we know that for some $\tau_0 > 0$, $t_\gamma(\tau_0) < t_1 e^{-\pi}$. Then

$$(t_1, x_1) \subset I^+(t_\gamma(\tau_0), x_\gamma(\tau_0)) \subset I^+(\gamma(0))$$

and so

$$\mathcal{M} \subset I^+(\gamma(0)).$$

Since $I^+(\partial\mathcal{M}) \subset \mathcal{M}$, we are left to conclude that

$$I^+(\gamma(0)) = \mathcal{M}.$$

So if there are at least two points in $\partial\mathcal{M}$ which can be connected in a time-like way to $\mathcal{M}$, then these points will have the same future, so that $\mathcal{M}'$ is not future distinguishing.

Now assume that there is an open set $U \in \partial\mathcal{M}$ in which every point can be connected to a point $y \in \mathcal{M}$ by a time-like curve. Given two points $x_1, x_2 \in U$, and a time-like path γ between x_2 to y so $\gamma(0) = x_2$, we can construct a series of time-like paths from x_1 to points arbitrarily close to x_2. To do this, note that since $I^+(x_1) = \mathcal{M}$, this means there is a time-like path from x_1 to $\gamma(t)$ for any $t > 0$. The limit of these curves will then be a null curve in $\partial\mathcal{M}$ between x_1 and x_2 (it is null because $I^+(U) = \mathcal{M}$ and so U does not contain any time-like curves.) It is futhermore a future pointing null curve, since it is the limit of future pointing time-like curves. But since our choice of x_1 and x_2 were arbitrary, there will also be a time-oriented null curve between x_2 and x_1, and so combining these curves gives us a closed null loop in $\partial\mathcal{M}$. In particular, any extension of the metric which would preserve the translation invariance of the metric is has a null loop.

Task 4

Book Review: Quantum Field Theory by Thanu Padmanabhan

Quantum Field Theory by Thanu Padmanabhan aims to provide a first introduction to QFT. The author's approach to the topic is to deriving QFT by evaluating the relativistic path integral, a method not found in other textbooks. A broad approach to field theory is taken, with non-perturbative aspects brought to the foreground and a Wilsonian approach adopted throughout. With only 280 pages, the text is serves as a first introduction to QFT, from which reader can progress to more advanced texts.

A chief aim of the author is to introduce QFT logically and consistently, and for the most part is successful in this aim. The author begins with the path-integral for non-relativistic particles, and progresses then to the path-integral for relativistic particles. This makes for a transparent approach to field theory, so that students are able to understand the motivations behind the formal manipulations. Chapter 1 is successful in showing the origin of fields from this process. Chapter 2 is more difficult to follow, and while the author is attempting to motivate the path-integral for scalar fields through the discussion of section 2.1, a lot of the material would be easier to derive and understand if presented using the path-integral for scalar fields.

Readers not familiar with the many-paths formulation of quantum mechanics would struggle with the first chapter. The section is too short and not sufficiently motivated to be transparent for a first time introduction. The connection between path-integrals and statistical mechanics could also be elaborated on, especially considering that this connection underpins the section on the Unruh effect. Similarly, the aside about occupation number basis is too short to do justice to the topic if students are not already familiar with the material.

The material in chapter 3 felt a bit disjointed. The discussion of classical electromagnetism would make more sense after the canonical quantization of scalar field theory, allowing readers to view the entire quantization process for scalar fields, and then afterwards view the process for electromagnetism with gauge symmetry. Noether's theorem and Goldstone's theorem could be emphasized more; it would also be good to give their names in the textbook so that students can find more information about these central results. The introduction of gauge fields by attempting to make global symmetries local is unmotivated and therefore difficult to understand. Since the textbook does not cover Yang-Mills theories, it could be simpler to start with the electromagnetic Lagrangian and demonstrating that the gauge symmetry can be used to avoid negative normed states.

In chapter 4, the Wilsonian approach to field theory is well explained, and its use from the onset makes the divergences and the renormalization process much less mysterious than in other textbooks. The effective action is then used to study non-perturbative phenomena, including a deriving the Euler-Heisenberg Lagrangian and the Schwinger effect. This approach is different from other texts, and aids the conceptual development of field theory. Running coupling constants are introduced in this context. One result missing here is the decoupling theorem, which is touched upon but not explained as a fully general process for removing heavy fields.

The derivation of the Feynman rules is concise yet easy to follow because the necessary groundwork has already been laid. One issue with this section is that the connection between Feynman diagrams and the physically measurable scattering amplitudes is not made very explicit. It would be useful if at least one full cross-section was included somewhere, even just as an aside, since it connects field theory to the data from high energy experiments. Decay rates are not discussed, and the optical theorem only briefly, even though particle decays are easy to treat and also important to understand. $\lambda\phi^4$ theory is used to illustrate the cancellation of divergences to one-loop in a renormalizable theory.

In chapter 5, the gamma matrices are first introduced through a discussion of the Pauli matrices in NRQM. This is a very easy to understand route to spinors, since they are first introduced in a familiar context. It also shows that spin, and more specifically, the factor of 2 in the gyromagnetic ratio, can be fully understood in NRQM, contrary to the claims of other texts.

The section on the Lorentz group representations is good, however, it focuses a bit much on the Dirac spinor. Perhaps an explanation of where the vector and tensor representations come from would help readers to better understand the material; a table of the low dimensional representations and their physical applications could be useful.

The book finishes with a detailed discussion of QED to one loop. However, the coverage of QED omits a number of important topics; there is no calculation of tree-level processes, bound states, or infrared divergences in the text for instance.

There are problems throughout the book in the margins. Most of these are fairly straightforward calculations which fill in gaps in the main text. Eighteen longer problems are included at the end of the book, though most of these relate to the last two chapters of the text. While there are a few more involved questions, as a whole they are not particularly challenging. A few of them present material not covered in the main text, such as Compton scattering and an anomaly calculation. The book could however do with many more exercises of these kinds, both because they could be used to include otherwise omitted material, and because they would be useful meatier problems for self-study. It would also be helpful if these were included directly after the relevant chapter, rather than right at the end of the book.

Overall the text serves as a good first introduction to the basics of QFT. It includes various conceptually important topics missed by textbooks more focused on calculations. The book would serve well as a general introduction to QFT, suitable for people with a more diverse interests than just Feynman diagram calculations. However it misses important calculational aspects of QFT, leaving the presentation of QED incomplete. Because of this, the book would need to be supplemented if being used to teach in a course oriented towards particle physics.

Task 5

Exercises from Hughston and Tod

This section comprises of solutions to problems in *An Introduction to General Relativity* by Hughston and Tod.

5.1 Tensor Analysis and Covariant Differentiation

These exercises are taken from Chapter 4 and 5 of Hughston and Tod.

4.1. If x^a and x'^a are coordinates for some region of M, then

$$\frac{\partial x'^a}{\partial x^p}\frac{\partial^2 x^p}{\partial x'^b \partial x'^c} = -\frac{\partial x^q}{\partial x'^b}\frac{\partial x^r}{\partial x'^c}\frac{\partial^2 x'^a}{\partial x^q \partial x^r}.$$

Solution: By differentiating the identity

$$\frac{\partial x'^a}{\partial x^p}\frac{\partial x^p}{\partial x'^b} = \delta'^a_{\prime b},$$

we find that

$$\frac{\partial}{\partial x'^c}\left(\frac{\partial x'^a}{\partial x^p}\frac{\partial x^p}{\partial x'^b}\right) = 0.$$

Using the Leibniz rule, we find

$$\frac{\partial}{\partial x'^c}\left(\frac{\partial x'^a}{\partial x^p}\right)\frac{\partial x^p}{\partial x'^b} + \frac{\partial x'^a}{\partial x^p}\frac{\partial}{\partial x'^c}\left(\frac{\partial x^p}{\partial x'^b}\right) = 0$$

and so

$$\frac{\partial x'^a}{\partial x^p}\frac{\partial^2 x^p}{\partial x'^b \partial x'^c} = -\frac{\partial x^p}{\partial x'^b}\frac{\partial}{\partial x'^c}\left(\frac{\partial x'^a}{\partial x^p}\right)$$

$$= -\frac{\partial x^q}{\partial x'^b}\frac{\partial x^r}{\partial x'^c}\frac{\partial}{\partial x^r}\left(\frac{\partial x'^a}{\partial x^q}\right) = -\frac{\partial x^q}{\partial x'^b}\frac{\partial x^r}{\partial x'^c}\frac{\partial^2 x'^a}{\partial x^q \partial x^r}.$$

4.2: If A_a is a covariant vector field, then show $\partial_{[a}A_{b]}$ is a tensor of valence $[0,2]$. Likewise, show if B_{ab} is antisymmetric, the $\partial_{[a}B_{bc]}$.

Solution: Under a coordinate transformation from coordinates x^a to x'^a, we find that

$$2\partial_{[a}A_{b]} = \partial_a A_b - \partial_b A_a = \frac{\partial x'^a}{\partial x^a}\frac{\partial}{\partial x'^a}\left(\frac{\partial x'^b}{\partial x^b}A'_b\right) - \frac{\partial x'^b}{\partial x^b}\frac{\partial}{\partial x'^b}\left(\frac{\partial x'^a}{\partial x^a}A'_a\right)$$

$$= \frac{\partial x'^a}{\partial x^a}\frac{\partial x'^b}{\partial x^b}\frac{\partial A'_b}{\partial x'^a} + A'_b\frac{\partial^2 x'^b}{\partial x^a \partial x^b} - \frac{\partial x'^b}{\partial x^b}\frac{\partial x'^a}{\partial x^a}\frac{\partial A'_a}{\partial x'^b} - A'_a\frac{\partial^2 x'^a}{\partial x^b \partial x^a}$$

$$= \frac{\partial x'^a}{\partial x^a}\frac{\partial x'^b}{\partial x^b}\left(\frac{\partial A'_b}{\partial x'^a} - \frac{\partial A'_a}{\partial x'^b}\right) + A'_b\left(\frac{\partial^2 x'^a}{\partial x^a \partial x^b} - \frac{\partial^2 x'^a}{\partial x^b \partial x^a}\right)$$

$$= 2\frac{\partial x'^a}{\partial x^a}\frac{\partial x'^b}{\partial x^b}\partial_{['a}A'_{b]}.$$

Hence $\partial_{[a}A_{b]}$ transforms as a tensor of valence $(0,2)$.

The set of tensors of types $(0,2)$ is spanned by $e_a e_b$ for basis $e_a = \frac{\partial}{\partial x^a}$ of T_p^* (see for instance pg 18 of Hawking and Ellis). In particular, any antisymmetric tensor C_{ab} can be decomposed into the sum of tensors $e_a e_b - e_b e_a$. So all we need to show is that the antisymmetrization of

$$\partial_a(\phi(e_b e_c - e_c e_b))$$

transforms as a tensor, for arbitrary scalar functions ϕ. As $\partial_{[a}C_{bc]}$ is the sum of terms of this form, this would then prove that C_{ab} was a tensor.

We can calculate

$$\partial_a(\phi(e_b e_c - e_c e_b)) = (\partial_a\phi)(e_b e_c - e_c e_b) + \phi\partial_a(e_b e_c - e_c e_b)$$

The first term is a tensor, since it is the tensor product of two tensors. So all that's left to prove is that the antisymmetrization of $\partial_a(e_b e_c - e_c e_b)$ is a tensor, as this would then show that the second term is tensor. We compute

$$\partial_{[a}e_b e_{c]} - \partial_{[a}e_c e_{b]} = e_{[b}\partial_a e_{c]} - e_{[c}\partial_a e_{b]} + (\partial_{[a}e_b)e_{c]} - (\partial_{[a}e_c)e_{b]}$$

$$= e_{[b}\partial_a e_{c]} - e_{[b}\partial_c e_{a]} + (\partial_{[a}e_b)e_{c]} - (\partial_{[b}e_a)e_{c]}$$

$$= \text{antisymmetrization of } 2e_b(\partial_{[a}e_{c]}) + 2(\partial_{[a}e_{b]})e_c.$$

We showed above that $\partial_{[a}e_{c]}$ is a tensor, and so $2e_b(\partial_{[a}e_{c]}) + 2(\partial_{[a}e_{b]})e_c$ is also a tensor. This means its antisymmetrization is a tensor, and so $\partial_{[a}e_b e_{c]} - \partial_{[a}e_c e_{b]}$ is a tensor, completing the proof.

4.3: Suppose P^a and Q^a are contravariant vector fields, then show that

$$P^a\partial_a Q^b - Q^a\partial_a P^b$$

is also a contravariant vector field.

Solution: Under a coordinate change from x^b to x'^b, we find that

$$P^a\partial_a Q^b - Q^a\partial_a P^b = P^a\partial_a\left(\frac{\partial x^b}{\partial x'^b}Q'^b\right) - Q^a\partial_a\left(\frac{\partial x^b}{\partial x'^b}P'^b\right)$$

$$= \frac{\partial x^b}{\partial x'^b}P^a\partial_a Q'^b + Q'^b P^a\left(\frac{\partial x'^a}{\partial x^a}\frac{\partial x^b}{\partial x'^a x'^b}\right) - \frac{\partial x^b}{\partial x'^b}Q^a\partial_a P'^b - Q^a P'^b\left(\frac{\partial x'^a}{\partial x^a}\frac{\partial x^b}{\partial x'^a x'^b}\right)$$

$$= \frac{\partial x^b}{\partial x'^b}(P^a \partial_a Q'^b - Q^a \partial_a P'^b).$$

Hence $P^a \partial_a Q^b - Q^a \partial_a P^b$ transforms as a contravariant vector field.

4.4 Show that if P^a and S_a are contravariant and covariant vector fields respectively, then

$$T_b = P^a \partial_a S_b + S_a \partial_b P^a$$

is also covariant.
Solution: We simply calculate

$$T_b = P^a \partial_a S_b + S_a \partial_b P^a = P^a \partial_a S_b + \partial_b (S_a P^a) - P^a \partial_b S_a$$

$$= P^a (\partial_a S_b - \partial_b S_a) + \partial_b (S_a P^a).$$

The first term is covariant, since we have already proved that $\partial_a S_b - \partial_b S_a$ is a tensor of rank $(0, 2)$. The second term is also covariant, as it is the derivative of a scalar function. So T_b is covariant.

5.1 If W_a has type $(0, 1)$, verify that $\partial_b W_a$ has type $(0, 2)$.
Solution: Using the modified transformation laws given at the start of chapter 5, we find that under a coordinate transformation,

$$\nabla_{'b} W_{'a} = \partial_{'b} W_{'a} - \Gamma'^c_{'b'a} W_{'c}$$

$$= \frac{\partial x^b}{\partial x'^b} \partial_b \left(\frac{\partial x^a}{\partial x'^a} W_a \right) - \frac{\partial x'^c}{\partial x^p} \frac{\partial x^q}{\partial x'^b} \frac{\partial x^r}{\partial x'^a} \Gamma^p_{qr} \frac{\partial x^c}{\partial x'^c} W_c - \frac{\partial x'^c}{\partial x^p} \frac{\partial^2 x^p}{\partial x'^b \partial x'^a} \frac{\partial x^c}{\partial x'^c} W_c$$

$$= \frac{\partial x^b}{\partial x'^b} \frac{\partial x^a}{\partial x'^a} \partial_b W_a - \frac{\partial x^b}{\partial x'^b} \frac{\partial x^a}{\partial x'^a} \Gamma^c_{ba} W_c + W_a \frac{\partial x^b}{\partial x'^b} \frac{\partial^2 x^a}{\partial x^b \partial x'^a} - \frac{\partial^2 x^c}{\partial x'^b \partial x'^a} W_c$$

$$= \frac{\partial x^b}{\partial x'^b} \frac{\partial x^a}{\partial x'^a} (\partial_b W_a - \Gamma^c_{ba} W_c) = \frac{\partial x^b}{\partial x'^b} \frac{\partial x^a}{\partial x'^a} \nabla_b W_a.$$

5.2 If the connection satisfies

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) V_c = T^d_{ab} \nabla_d V_c$$

for every smooth vector field V_c, then

$$\nabla_{[a} T^d_{bc]} + T^r_{[ab} T^d_{c]r} = 0.$$

Solution: We shall explore the case where $V_c = \nabla_c \phi$ for some scalar field ϕ. In this case we find that

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) \nabla_c \phi = T^d_{ab} \nabla_d \nabla_c \phi.$$

Since from the definition of torsion,

$$(\nabla_c \nabla_d + T^r_{dc} \nabla_r) \phi = \nabla_d \nabla_c \phi$$

we can deduce that

$$\nabla_a \nabla_b \nabla_c \phi - \nabla_b \nabla_a \nabla_c \phi = T^d_{ab} (\nabla_c \nabla_d + T^r_{dc} \nabla_r) \phi = T^d_{ab} \nabla_c \nabla_d \phi + T^d_{ab} T^r_{dc} \nabla_r \phi$$

$$= \nabla_c (\nabla_a \nabla_b - \nabla_b \nabla_a) \phi - (\nabla_c T^d_{ab}) \nabla_d \phi - T^r_{ab} T^d_{cr} \nabla_d \phi.$$

Rearranging, we find that

$$(\nabla_a\nabla_b\nabla_c - \nabla_b\nabla_a\nabla_c + \nabla_c\nabla_b\nabla_a - \nabla_c\nabla_a\nabla_b)\phi = -(\nabla_c T^d_{ab})\nabla_d\phi - T^r_{ab}T^d_{cr}\nabla_d\phi$$

Since the antisymmetrisation of the left-hand side is zero, we have that for every scalar field ϕ,

$$(\nabla_{[c}T^d_{ab]})\nabla_d\phi + T^r_{[ab}T^d_{c]r}\nabla_d\phi = 0.$$

Every covariant vector V_d is the gradient of some scalar vector field, and so we conclude that

$$(\nabla_{[c}T^d_{ab]} + T^r_{[ab}T^d_{c]r})V_d = 0$$

for every covariant vector V_d. This is only possible if

$$\nabla_{[c}T^d_{ab]} + T^r_{[ab}T^d_{c]r} = 0.$$

5.2 The Riemann Tensor and Riemannian Geometry

These exercises are taken from Chapter 6 and 7 of Hughston and Tod.

6.1. If $F_{\alpha\beta}$ is antisymmetric, then for torsion free ∇_a, we have

$$\nabla_{[a}\nabla_{b]}F_{cd} = \alpha R_{ab[c}{}^eF_{d]e}$$

where α is a constant to be computed.
Solution: Using equation (6.2.4), we have

$$\begin{aligned}\nabla_{[a}\nabla_{b]}F_{cd} &= -\frac{1}{2}R_{abc}{}^rF_{rd} - \frac{1}{2}R_{abd}{}^rF_{cr}\\ &= \frac{1}{2}R_{abc}{}^rF_{dr} - \frac{1}{2}R_{abd}{}^rF_{cr}\\ &= R_{ab[c}{}^rF_{d]r},\end{aligned}$$

so $\alpha = 1$.

6.2. If ∇_a and $\tilde{\nabla}_a$ are connections with

$$\tilde{\nabla}_aV^b = \nabla_aV^b + Q^b_{ac}V^c$$

for all vector fields V^b, show that

$$\frac{1}{2}(\tilde{R}_{abc}{}^d - R_{abc}{}^d) = \nabla_{[a}Q^d_{b]c} + Q^d_{r[a}Q^r_{b]c}$$

Solution: If $\nabla_aV^b = \partial_aV^b + \Gamma^b_{ac}V^c$, then $\tilde{\nabla}_aV^b = (\partial_aV^b + (\Gamma^b_{ac} + Q^b_{ac})V^c$. We can now apply (6.1.6) to find

$$\frac{1}{2}R_{abc}{}^d = \partial_{[a}\Gamma^d_{b]c} - \Gamma^p_{[a|c}\Gamma^d_{b]p}$$

$$\frac{1}{2}\tilde{R}_{abc}{}^{d} = \partial_{[a}(\Gamma^{d}_{b]c} + Q^{d}_{b]c}) - (\Gamma^{p}_{[a|c} + Q^{p}_{[a|c})(\Gamma^{d}_{b]p} + Q^{d}_{b]p})$$
$$= \partial_{[a}\Gamma^{d}_{b]c} - \Gamma^{p}_{[a|c}\Gamma^{d}_{b]p} + \partial_{[a}Q^{d}_{b]c} - Q^{p}_{[a|c}Q^{d}_{b]p} + Q^{p}_{[a|c}\Gamma^{d}_{b]p} - \Gamma^{p}_{[a|c}Q^{d}_{b]p}$$
$$= \frac{1}{2}R_{abc}{}^{d} + \partial_{[a}Q^{d}_{b]c} - \Gamma^{d}_{[a|p}Q^{p}_{b]c} - \Gamma^{p}_{[a|c}Q^{d}_{b]p} + Q^{d}_{[a|p}Q^{p}_{b]c}.$$

Now using the formula

$$\nabla_{[a}Q^{d}_{b]c} = \partial_{[a}Q^{d}_{b]c} - \Gamma^{d}_{[a|p}Q^{p}_{b]c} - \Gamma^{r}_{[ab]}Q^{d}_{rc} - \Gamma^{p}_{[a|c}Q^{d}_{b]p}$$

we find

$$\frac{1}{2}\tilde{R}_{abc}{}^{d} - \frac{1}{2}R_{abc}{}^{d} = \nabla_{[a}Q^{d}_{b]c} + \Gamma^{r}_{[ab]}Q^{d}_{rc} + Q^{d}_{[a|p}Q^{p}_{b]c}.$$

Although the question does not specify, as far as I can tell the question requires the two connections to be torsion free, as in this case,

$$\Gamma^{r}_{[ab]} = 0, \quad Q^{d}_{[a|p}Q^{p}_{b]c} = Q^{d}_{p[a}Q^{p}_{b]c}$$

and so

$$\frac{1}{2}\tilde{R}_{abc}{}^{d} - \frac{1}{2}R_{abc}{}^{d} = \nabla_{[a}Q^{d}_{b]c} + Q^{d}_{p[a}Q^{p}_{b]c}.$$

6.3. If ∇_a is torsion-free, show that

$$\nabla_{[a}R_{b]c} = -\frac{1}{2}\nabla_d R_{abc}{}^{d}.$$

Solution: We can expand the left-hand side of the equation to find that

$$\nabla_{[a}R_{b]c} = \nabla_{[a}R_{b]dc}{}^{d} = \frac{1}{2}(\nabla_a R_{bdc}{}^{d} - \nabla_b R_{adc}{}^{d})$$
$$= \frac{1}{2}(\nabla_a R_{bdc}{}^{d} + \nabla_b R_{dac}{}^{d}) = \frac{3}{2}\nabla_{[a}R_{bd]c}{}^{d} - \frac{1}{2}\nabla_d R_{abc}{}^{d}$$

where the final two equalities have used the symmetry $R_{abc}{}^{d} = -R_{bac}{}^{d}$. The Bianchi identity states that

$$\nabla_{[a}R_{bd]c}{}^{d} = 0,$$

and hence

$$\nabla_{[a}R_{b]c} = -\frac{1}{2}\nabla_d R_{abc}{}^{d}.$$

6.4. Show that if the torsion does not vanish then

$$R_{[abc]}{}^{d} + \nabla_{[a}T^{d}_{bc]} + T^{p}_{[ab}T^{d}_{c]p} = 0.$$

Solution: Using equation (6.2.1), we have

$$\nabla_{[a}\nabla_{b]}V_c = \frac{1}{2}T^{d}_{ab}\nabla_d V_c - \frac{1}{2}R_{abc}{}^{d}V_d.$$

so in particular if $V_c = \nabla_c \phi$ for some scalar field ϕ, then

$$\nabla_{[a}\nabla_{b]}\nabla_c\phi = \frac{1}{2}(T^d_{ab}\nabla_d - R_{abc}{}^d)\nabla_c\phi.$$

Since from the definition of torsion,

$$(\nabla_c\nabla_d + T^r_{dc}\nabla_r)\phi = \nabla_d\nabla_c\phi$$

we find that

$$\begin{aligned}\nabla_{[a}\nabla_{b]}\nabla_c\phi &= \frac{1}{2}T^d_{ab}(\nabla_c\nabla_d + T^r_{dc}\nabla_r)\phi - \frac{1}{2}R_{abc}{}^d\nabla_d\phi \\ &= \nabla_c\nabla_{[a}\nabla_{b]}\phi - \frac{1}{2}(\nabla_c T^d_{ab} + T^r_{ab}T^d_{cr})\nabla_d\phi - \frac{1}{2}R_{abc}{}^d\nabla_d\phi.\end{aligned}$$

Therefore

$$\frac{1}{2}(\nabla_c T^d_{ab} + T^r_{ab}T^d_{cr})\nabla_d\phi + \frac{1}{2}R_{abc}{}^d\nabla_c\phi = \nabla_c\nabla_{[a}\nabla_{b]}\phi - \nabla_{[a}\nabla_{b]}\nabla_c\phi.$$

By permuting through a, b and c and noting that both T^d_{ab} and $R_{abc}{}^d$ are antisymmetric in ab, we hence find that

$$(\nabla_{[c}T^d_{ab]} + T^r_{[ab}T^d_{c]r} + R_{[abc]}{}^d)\nabla_d\phi = 0.$$

Since this holds for arbitrary $\nabla_d\phi$, and since any covariant vector is the derivative of some scalar function, this means that

$$\nabla_{[c}T^d_{ab]} + T^r_{[ab}T^d_{c]r} + R_{[abc]}{}^d = 0.$$

6.6 For an n-dimensional manifold with symmetric connection Γ^c_{ab}, define

$$P_{ij} = -\frac{n}{n^2-1}R_{ij} - \frac{1}{n^2-1}R_{ji}.$$

We say that M is projectively flat if the connection can be put into the form

$$\Gamma^c_{ab} = \phi_a\delta^c_b + \phi_b\delta^c_a.$$

Show that

1. $R_{jki}{}^h + \delta^h_k P_{ji} - \delta^h_j P_{ki} - (P_{kj} - P_{jk})\delta^h_i = 0$
2. $\nabla_k P_{ji} = \nabla_j P_{ki}$.

Furthermore show that if $\phi_a = \nabla_a\phi$ for some scalar field ϕ, then $P_{ij} = P_{ji}$. Show that if $n > 2$ the first equation implies the second equation.

Solution: Using (6.1.6), we have

$$\begin{aligned}\frac{1}{2}R_{abc}{}^d &= \partial_{[a}\Gamma^d_{b]c} - \Gamma^p_{[a|c|}\Gamma^d_{b]p} \\ &= \partial_{[a}\phi_{b]}\delta^d_c + \delta^d_{[b|}\partial_{|a]}\phi_c - (\phi_{[a|}\delta^p_c + \phi_c\delta^p_{[a|})(\phi_{b]}\delta^d_p + \phi_p\delta^d_{b]}) \\ &= \partial_{[a}\phi_{b]}\delta^d_c + \delta^d_{[b|}\partial_{|a]}\phi_c - \phi_{[a|}\phi_{b]}\delta^d_c - \phi_c\delta^d_{[a|}\phi_{b]} - \phi_{[a|}\phi_c\delta^d_{b]} - \phi_c\phi_{[a}\delta^d_{b]} \\ &= \partial_{[a}\phi_{b]}\delta^d_c + \delta^d_{[b|}\partial_{|a]}\phi_c - \phi_{[a|}\phi_{b]}\delta^d_c - \phi_{[a|}\phi_c\delta^d_{b]}.\end{aligned}$$

We hence can write

$$R_{abc}{}^d = (\partial_a\phi_b - \partial_b\phi_a)\delta^d_c + (\partial_a\phi_c - \phi_a\phi_c)\delta^d_b - (\partial_b\phi_c - \phi_b\phi_c)\delta^d_a$$

The Ricci tensor is then

$$R_{ij} = R_{ibj}{}^{b} = (\partial_i\phi_j - \partial_j\phi_i) + n(\partial_i\phi_j - \phi_i\phi_j) - (\partial_i\phi_j - \phi_i\phi_j)$$
$$= n\partial_i\phi_j - (n-1)\phi_i\phi_j - \partial_j\phi_i.$$

We can then calculate

$$P_{ij} = -\frac{n}{n^2-1}R_{ij} - \frac{1}{n^2-1}R_{ji}$$
$$= -\frac{n}{n^2-1}(n\partial_i\phi_j - (n-1)\phi_i\phi_j - \partial_j\phi_i) - \frac{1}{n^2-1}(n\partial_j\phi_i - (n-1)\phi_j\phi_i - \partial_i\phi_j)$$
$$= -\frac{1}{n^2-1}\left((n^2-1)\partial_i\phi_j - (n^2-1)\phi_i\phi_j\right) = \phi_i\phi_j - \partial_i\phi_j.$$

For the case where $\phi_i = \nabla_i\phi$, we find that

$$P_{ij} = \phi_i\phi_j - \partial_i\phi_j = \phi_i\phi_j - \partial_i\partial_j\phi = \phi_j\phi_i - \partial_j\partial_i\phi = P_{ji}$$

so in this specific situation, $P_{ij} = P_j i$.

To prove the first identity, we calculate

$$\delta^h_j P_{ki} + (P_{kj} + P_{jk})\delta^h_i - \delta^h_k P_{ji} = \delta^h_j(\phi_k\phi_i - \partial_k\phi_i) + (\partial_j\phi_k - \partial_k\phi_j)\delta^h_i - \delta^h_k(\phi_i\phi_j - \partial_j\phi_i) = R^h_{jki}$$

from which the first identity follows.

We now derive the second identity

$$\nabla_k P_{ji} = \partial_k(\phi_j\phi_i - \partial_j\phi_i) - \Gamma^h_{ki}(\phi_j\phi_h - \partial_j\phi_h) - \Gamma^h_{kj}(\phi_h\phi_i - \partial_h\phi_i)$$
$$= \partial_k(\phi_j\phi_i - \partial_j\phi_i) - (\phi_k\delta^h_i + \phi_i\delta^h_k)(\phi_j\phi_h - \partial_j\phi_h) - (\phi_k\delta^h_j + \phi_j\delta^h_k)(\phi_h\phi_i - \partial_h\phi_i)$$
$$= \partial_k(\phi_j\phi_i) - \partial_j\partial_k\phi_i - 4\phi_j\phi_i\phi_k + \phi_k\partial_j\phi_i + \phi_i\partial_j\phi_k + \phi_k\partial_j\phi_i + \phi_j\partial_k\phi_i$$
$$= \partial_k(\phi_j\phi_i) + \partial_j(\phi_k\phi_i) - \partial_j\partial_k\phi_i - 4\phi_j\phi_i\phi_k + \phi_k\partial_j\phi_i + \phi_j\partial_k\phi_i.$$

The last expression above is manifestly symmetric upon interchange of j and k, so

$$\nabla_k P_{ji} = \nabla_j P_{ki}.$$

If $n > 2$, we contract h with k in (1) to get

$$R_{ji} + nP_{ji} = P_{ij}.$$

Using this expression and the expression with j and i flipped, we can eliminate P_{ij} giving us

$$\nabla_k P_{ji} = \frac{1}{1-n-n^2}\nabla_k R_{ij}$$

The second expression now follows by taking the derivative of this expression and applying result of question 6.3:

$$\nabla_k P_{ji} = \frac{1}{1-n-n^2}\nabla_k R_{ij} = \frac{1}{1-n-n^2}\nabla_j R_{ki} = \nabla_j P_{ki}.$$

6.7 Let $\Delta_{ab} = 2\nabla_{[a}\nabla_{b]}$ and assume that the torsion vanishes, so $\nabla_{ab}\phi = 0$ for all scalar field.Show that

$$\Delta_{ab}(PQ) = P\Delta_{ab}Q + Q\Delta_{ab}P$$

for all tensors P and Q.
Solution: Using the Liebniz rule, we can expand

$$\Delta_{ab}(PQ) = 2\nabla_{[a}\nabla_{b]}PQ = 2\nabla_{[a}\left(P\nabla_{b]}Q + Q\nabla_{b]}P\right)$$

$$= 2P\nabla_{[a}\nabla_{b]}Q + 2Q\nabla_{[a}\nabla_{b]}P + 2(\nabla_{[a}P)(\nabla_{b]}Q) + 2(\nabla_{[a}Q)(\nabla_{b]}P)$$
$$= P\Delta_{ab}Q + Q\Delta_{ab}P + 2(\nabla_{[a}P)(\nabla_{b]}Q) - 2(\nabla_{[b}Q)(\nabla_{a]}P)$$
$$= P\Delta_{ab}Q + Q\Delta_{ab}P.$$

7.1 Verify directly that if $\Gamma^d_{bc} = \frac{1}{2}g^{da}(\partial_b g_{ca} + \partial_c g_{ba} - \partial_a g_{bc})$ then $\nabla_a g_{bc} = 0$.
Solution: We directly compute

$$\nabla_a g_{bc} = \partial_a g_{bc} - \Gamma^d_{ab}g_{dc} - \Gamma^d_{ac}g_{bd}$$

$$= \partial_a g_{bc} - \frac{1}{2}g^{dr}(\partial_a g_{br} + \partial_b g_{ar} - \partial_r g_{ab})g_{dc} - \frac{1}{2}g^{dr}(\partial_a g_{cr} + \partial_c g_{ar} - \partial_r g_{ac})g_{db}$$
$$= \partial_a g_{bc} - \frac{1}{2}\delta^r_c(\partial_a g_{br} + \partial_b g_{ar} - \partial_r g_{ab}) - \frac{1}{2}\delta^r_b(\partial_a g_{cr} + \partial_c g_{ar} - \partial_r g_{ac})$$
$$= \partial_a g_{bc} - \frac{1}{2}\left(\partial_a g_{bc} + \partial_b g_{ac} - \partial_c g_{ab} + \partial_a g_{cb} + \partial_c g_{ab} - \partial_b g_{ac}\right) = 0.$$

7.2 Show that if $\nabla_a g_{bc} = 0$, then $\nabla_a g^{bc} = 0$.
Solution: Let us use the Liebniz rule to compute

$$\nabla_a(g_{bc}g^{cd}) = (\nabla_a g_{bc})g^{cd} + g_{bc}\nabla_a g^{cd} = g_{bc}\nabla_a g^{cd}.$$

But since $g_{bc}g^{cd} = \delta^d_b$, we find that

$$\nabla_a(g_{bc}g^{cd}) = \nabla_a(\delta^d_b) = \partial_a\delta^d_b + \Gamma^d_{ar}\delta^r_b - \Gamma^r_{ab}\delta^d_r = 0.$$

Combining these two equations gives us

$$g_{bc}\nabla_a g^{cd} = 0$$

which then implies that $\nabla_a g^{cd} = 0$, since g_{bc} is nondegenerate.

7.3 Show that if $R_{ab} = 0$ then $\nabla^a R_{abcd} = 0$.
Solution: Beginning with the Bianchi identity

$$\nabla_e R_{abcd} + \nabla_a R_{becd} + \nabla_b R_{eacd} = 0$$

we can contract with g^{ea} to get

$$\nabla^a R_{abcd} + g^{ea}\nabla_a R_{becd} = g^{ea}\nabla_b R_{eacd} = \nabla_b R^a{}_{acd}.$$

Using the symmetries of the Riemann tensor, we obtain the identiy we sought:

$$2\nabla^a R_{abcd} = \nabla_b R_{ca}{}^a{}_d = \nabla_b R^a_{cad} = \nabla R_{cd} = 0.$$

7.4 Show that if ξ_a satisfies Killing's equation $\nabla_{(a}\xi_{b)} = 0$, then the tensor $F_{ab} = \nabla_a \xi_b$ satisfies Maxwell's vacuum equations, $\nabla^a F_{ab} = 0$ and $\nabla_{[a}F_{bc]} = 0$.
Solution: We can using Killing's equation to write

$$F_{ab} = \nabla_a \xi_b = -\nabla_b \xi_a = \nabla_{[a}\xi_{b]},$$

so F_{ab} is antisymmetric.

To derive the first Maxwell equation, we can manipulate

$$\nabla^a F_{ab} = \nabla^a \nabla_a \xi_b = -\nabla^a \nabla_b \xi_a = -\nabla_b(\nabla^a \xi_a)$$

$$= -\nabla_b(g^{ra}\nabla_r \xi_a) = \nabla_b(g^{ra}\nabla_a \xi_r) = \nabla_b(\nabla^a \xi_a) = -\nabla^a F_{ab}$$

and so

$$\nabla^a F_{ab} = 0.$$

For the second Maxwell equation, we use

$$\nabla_{[a}F_{bc]} = \frac{1}{3}(\nabla_a \nabla_b \xi_c + \nabla_b \nabla_c \xi_a + \nabla_c \nabla_a \xi_b) = \frac{1}{3}(\nabla_a \nabla_b \xi_c - \nabla_b \nabla_a \xi_c - \nabla_a \nabla_b \xi_c)$$

$$= -\frac{1}{3}\nabla_a \nabla_b \xi_c.$$

Since

$$\nabla_{[a}F_{bc]} = -\nabla_{[b}F_{ac} = \frac{1}{3}\nabla_b \nabla_a \xi_c = \frac{1}{3}\nabla_a \nabla_b \xi_c = -\nabla_{[a}F_{bc]},$$

we conclude that

$$\nabla_{[a}F_{bc]} = 0.$$

7.9. We say that the Ricci tensor is non-degenerate if it has everywhere non-vanishing determinant. Show that if a manifold has non-degenerate Ricci tensor, then there cannot exist non-trivial vector fields K_b satisfying $\nabla_a K_b = 0$.
Solution: Let K_a be a non-trivial vector field satisfying $\nabla_a K_b = 0$. Using equation (6.2.1),

$$\nabla_{[a}\nabla_{b]}K^c = \frac{1}{2}R_{abd}{}^c K^d$$

since metric connections are torsion free. Since

$$\nabla_{[a}\nabla_{b]}K^c = (\nabla_{[a}\nabla_{b]})(g^{cr}K_r) = \nabla_{[a}(g^{cr}\nabla_{b]}K_r + K_r \nabla_{b]}g^{cr}) = 0$$

we find that

$$R_{abd}{}^c K^d = 0$$

and hence that

$$R_{ad}K^d = R_{acd}{}^c K^d = 0.$$

This would then imply that R_{ad} was degenerate, as it maps a non-trivial vector K^d to zero. So we find that if R_{ad} is non-degenerate everywhere, then no such vector field K_a can exist.

7.10. Let $g = \det(g_{ab})$. Show that for any vector field V^a we have $\nabla_a V^a = (-g)^{-1/2}\partial_a(V^a(-g)^{1/2})$ if g_{ab} has Lorentzian signature.
Solution: We can expand the right-hand side

$$(-g)^{-1/2}\partial_a(V^a(-g)^{1/2}) = \partial_a V^a + (-g)^{-1/2}V^a\partial_a(-g)^{1/2}$$

$$= \partial_a V^a + \frac{1}{2}V^a g^{-1}\partial_a g = \partial_a V^a + \frac{1}{2}V^a\partial_a \ln g$$

$$= \partial_a V^a + \frac{1}{2}V^a\partial_a \mathrm{Tr}(\ln g_{cd}) = \partial_a V^a + \frac{1}{2}V^a\mathrm{Tr}(g^{cd}\partial_a g_{de})$$

$$= \partial_a V^a + \frac{1}{2}V^a g^{cd}\partial_a g_{cd}.$$

Now expanding the left-hand side, we find that

$$\nabla_a V^a = \partial_a V^a + \frac{1}{2}\Gamma^a_{ab}V^b = \partial_a V^a + \frac{1}{2}g^{ad}(\partial_a g_{db} + \partial_b g_{ad} - \partial_d g_{ab})V^b$$

$$= \partial_a V^a + \frac{1}{2}(\partial^a g_{ab} + \partial_b g_{ad} - \partial^a g_{ab})V^b = \partial_a V^a + \frac{1}{2}V^a g^{cd}\partial_a g_{cd} = (-g)^{-1/2}\partial_a(V^a(-g)^{1/2}),$$

which is what we set out to prove.

7.11. Let Y_{ab} be skew-symmetric satisfying $\nabla_{(a}Y_{b)c} = 0$. Show that if $R_{ab} = 0$, then $F_{ab} = R_{abcd}Y^{cd}$ satisfies Maxwell's vacuum equations, $\nabla^a F_{ab} = 0$ and $\nabla_{[a}F_{bc]} = 0$.
Solution:

To derive the first Maxwell equation, we can manipulate

$$\nabla^a F_{ab} = \nabla^a(R_{abcd}Y^{cd}) = Y^{cd}\nabla^a R_{abcd} + R_{abcd}\nabla^a Y^{cd}.$$

From the result of question 7.2, we know that the first term is 0 because $R_{ab} = 0$. Since $\nabla_{(a}Y_{b)c} = 0$, this means that $\nabla^a Y^{cd} = -\nabla^c Y^{ad}$. So we find that

$$\nabla^a F_{ab} = R_{abcd}\nabla^a Y^{cd} = -R_{abcd}\nabla^c Y^{ad} = -R_{cbad}\nabla^a Y^{cd}$$

where we have swapped the dummy indices a and c to get the last equality. Using the identity $R_{[abc]d} = 0$, we find that

$$R_{abcd}\nabla^a Y^{cd} = -R_{cbad}\nabla^a Y^{cd} = R_{bacd}\nabla^a Y^{cd} + R_{acbd}\nabla^a Y^{cd} = -R_{abcd}\nabla^a Y^{cd} + R_{bdac}\nabla^a Y^{cd}$$

and so

$$2R_{abcd}\nabla^a Y^{cd} = R_{bdac}\nabla^a Y^{cd} = -R_{dbac}\nabla^a Y^{cd} = -R_{abcd}\nabla^c Y^{da} = R_{abcd}\nabla^c Y^{ad} = -R_{abcd}\nabla^a Y^{cd}.$$

We have found that

$$3R_{abcd}\nabla^a Y^{cd} = 0$$

and as a result,

$$\nabla^a F_{ab} = 0.$$

For the second Maxwell equation, we hence can write

$$\nabla_{[a}F_{bc]} = \nabla_{[a}(R_{bc]de}Y^{de}) = Y^{de}\nabla_{[a}R_{bc]de} + R_{[bc|de}\nabla_{|a]}Y^{de}.$$

The first term is zero by the Bianchi identity, so

$$\nabla_{[a}F_{bc]} = R_{[bc|de}\nabla_{|a]}Y^{de} = g_{f[a}R_{bc]de}\nabla^f Y^{de} = -g_{f[a}R_{bc]de}\nabla^d Y^{fe}$$

$$= R_{[bc|de}\nabla_{|a]}Y^{de} = R_{bcde}\nabla_a Y^{de} + R_{abde}\nabla_c Y^{de} + R_{cade}\nabla_b Y^{de}.$$

I could not work out how to proceed from here.

7.13. Let $\tilde{\nabla}_a$ be a torsion-free connection, and define a new connection ∇_a such that

$$\nabla_a V^b = \tilde{\nabla}_a V^b - \frac{1}{2}T^b_{ac}V^c$$

with $T^b_{ac} = -T^b_{ca}$. Let $\Omega_{ab} = -\Omega_{ba}$ be a non-degenerate two-form with an inverse $\hat{\Omega}^{ab}$ so that $\Omega_{ab}\hat{\Omega}^{bc} = \delta^c_a$. Show that there is a unique choice for T^b_{ac} such that $\nabla_a\Omega_{bc} = 0$.
Solution: In order for $\nabla_a\Omega_{bc} = 0$, we require that

$$\nabla_a\Omega_{bc} = \tilde{\nabla}_a\Omega_{bc} + \frac{1}{2}T^d_{ab}\Omega_{dc} + \frac{1}{2}T^d_{ac}\Omega_{bd} = 0$$

Cycling through the indices, we get

$$\tilde{\nabla}_b\Omega_{ca} + \frac{1}{2}T^d_{bc}\Omega_{da} + \frac{1}{2}T^d_{ba}\Omega_{cd} = 0$$

$$\tilde{\nabla}_c\Omega_{ab} + \frac{1}{2}T^d_{ca}\Omega_{db} + \frac{1}{2}T^d_{cb}\Omega_{ad} = 0.$$

Now subtracting the second and third term from the first, we get

$$\tilde{\nabla}_a\Omega_{bc} - \tilde{\nabla}_b\Omega_{ca} - \tilde{\nabla}_c\Omega_{ab} = \frac{1}{2}T^d_{bc}\Omega_{da} + \frac{1}{2}T^d_{ba}\Omega_{cd} + \frac{1}{2}T^d_{ca}\Omega_{db} + \frac{1}{2}T^d_{cb}\Omega_{ad} - \frac{1}{2}T^d_{ab}\Omega_{dc} - \frac{1}{2}T^d_{ac}\Omega_{bd}.$$

We now can use the symmetries $T^d_{cb} = -T^d_{bc}$ and $\Omega_{ab} = -\Omega_{ba}$ to find that

$$\frac{1}{2}T^d_{bc}\Omega_{da} + \frac{1}{2}T^d_{ba}\Omega_{cd} + \frac{1}{2}T^d_{ca}\Omega_{db} + \frac{1}{2}T^d_{cb}\Omega_{ad} - \frac{1}{2}T^d_{ab}\Omega_{dc} - \frac{1}{2}T^d_{ac}\Omega_{bd}$$

$$= \frac{1}{2}T^d_{bc}\Omega_{da} + \frac{1}{2}T^d_{ba}\Omega_{cd} + \frac{1}{2}T^d_{ca}\Omega_{db} + \frac{1}{2}T^d_{ba}\Omega_{da} - \frac{1}{2}T^d_{ba}\Omega_{cd} - \frac{1}{2}T^d_{ca}\Omega_{db} = T^d_{bc}\Omega_{da}$$

and hence

$$\tilde{\nabla}_a\Omega_{bc} - \tilde{\nabla}_b\Omega_{ca} - \tilde{\nabla}_c\Omega_{ab} = T^d_{bc}\Omega_{da}.$$

Multiplying both sides by $\hat{\Omega}^{ae}$, we find that

$$\hat{\Omega}^{ae}(\tilde{\nabla}_a\Omega_{bc} - \tilde{\nabla}_b\Omega_{ca} - \tilde{\nabla}_c\Omega_{ab}) = T^d_{bc}\Omega_{da}\hat{\Omega}^{ae} = T^e_{bc}$$

which uniquely specifies T^e_{bc}.

Now all we need to do is to verify that this choice of T^a_{bc} always gives $\nabla_a \Omega_{bc} = 0$. We simply calcualte

$$\nabla_a \Omega_{bc} = \tilde{\nabla}_a \Omega_{bc} + \frac{1}{2} T^d_{ab} \Omega_{dc} + \frac{1}{2} T^d_{ac} \Omega_{bd}$$

$$= \tilde{\nabla}_a \Omega_{bc} + \frac{1}{2} \left(\hat{\Omega}^{ed} (\tilde{\nabla}_e \Omega_{ab} - \tilde{\nabla}_a \Omega_{be} - \tilde{\nabla}_b \Omega_{ea}) \right) \Omega_{dc} + \frac{1}{2} \left(\hat{\Omega}^{ed} (\tilde{\nabla}_e \Omega_{ac} - \tilde{\nabla}_a \Omega_{ce} - \tilde{\nabla}_c \Omega_{ea}) \right) \Omega_{bd}$$

$$= \tilde{\nabla}_a \Omega_{bc} + \frac{1}{2} \delta^e_c (\tilde{\nabla}_e \Omega_{ab} - \tilde{\nabla}_a \Omega_{be} - \tilde{\nabla}_b \Omega_{ea}) - \frac{1}{2} \delta^e_b (\tilde{\nabla}_e \Omega_{ac} - \tilde{\nabla}_a \Omega_{ce} - \tilde{\nabla}_c \Omega_{ea})$$

$$= \tilde{\nabla}_a \Omega_{bc} + \frac{1}{2} (\tilde{\nabla}_c \Omega_{ab} - \tilde{\nabla}_a \Omega_{bc} - \tilde{\nabla}_b \Omega_{ca}) - \frac{1}{2} (\tilde{\nabla}_b \Omega_{ac} - \tilde{\nabla}_a \Omega_{cb} - \tilde{\nabla}_c \Omega_{ba})$$

$$= \tilde{\nabla}_a \Omega_{bc} + \frac{1}{2} (\tilde{\nabla}_c \Omega_{ab} - \tilde{\nabla}_a \Omega_{bc} + \tilde{\nabla}_b \Omega_{ca}) - \frac{1}{2} (\tilde{\nabla}_b \Omega_{ca} - \tilde{\nabla}_a \Omega_{bc} - \tilde{\nabla}_c \Omega_{ab}) = 0.$$

5.3 Geodesics

These exercises are taken from Chapter 9 of Hughston and Tod.

9.1. Show that the geodesic equation can be put in Lagrangian form:

$$\frac{d}{ds} \frac{\partial L}{\partial \dot{x}^a} = \frac{\partial L}{\partial x^a}$$

with Lagrangian $L = \frac{1}{2} g_{ab} \dot{x}^a \dot{x}^b$.
Solution: Starting with our Lagrangian, we can calculate the Euler-Lagrange equations

$$\frac{d}{ds} \frac{\partial L}{\partial \dot{x}^a} - \frac{\partial L}{\partial x^a} = \frac{d}{ds} (g_{ab} \dot{x}^b) - \frac{1}{2} \frac{\partial g_{bc}}{\partial x^a} \dot{x}^b \dot{x}^c$$

$$= g_{ab} \ddot{x}^b + \frac{\partial g_{ab}}{\partial x^c} \dot{x}^b \dot{x}^c - \frac{1}{2} \frac{\partial g_{bc}}{\partial x^a} \dot{x}^b \dot{x}^c$$

$$= g_{ab} \ddot{x}^b + \frac{1}{2} \left(\partial_c g_{ab} + \partial_b g_{ac} - \partial_a g_{bc} \right) \dot{x}^b \dot{x}^c = 0.$$

Multiplying both sides g^{ad}, we find that

$$\ddot{x}^d + \frac{1}{2} g^{ad} \left(\partial_c g_{ab} + \partial_b g_{ac} - \partial_a g_{bc} \right) \dot{x}^b \dot{x}^c = \ddot{x}^d + \Gamma^d_{bc} \dot{x}^b \dot{x}^c = 0.$$

The left-hand equality is the geodesic equation.

9.2. Suppose a tensor T^{ab} is of form $T^{ab} = \rho u^a u^b$ where ρ is a scalar field and $u^a u_a = 1$. If $\nabla_a T^{ab} = 0$ show that $\nabla_a (\rho u^a) = 0$ and that $(u^a \nabla_a u^b) u^c = (u^a \nabla_a u^c) u^b$
Solution: Differentiate the identity $u^a u_a = 1$, we find that

$$\nabla_b (u_a u^a) = u_a \nabla_b u^a + u^a \nabla_b u_a = 2 u_a \nabla_b u^a = 0.$$

Now let us expand

$$\nabla_a T^{ab} = \nabla_a (\rho u^a u^b) = u^b \nabla_a (\rho u^a) + \rho u^a \nabla_a u^b$$

and so, contracting both sides by u_b, we find

$$\nabla_a(\rho u^a) + \rho u^a u_b \nabla_a u^b = \nabla_a(\rho u^a) = 0.$$

This then implies that

$$\nabla_a T^{ab} = u^b \nabla_a(\rho u^a) + \rho u^a \nabla_a u^b = \rho u^a \nabla_a u^b = 0,$$

so that $u^a \nabla_a u^b = 0$. A simple consequence of this is that $(u^a \nabla_a u^b) u^c = (u^a \nabla_a u^c) u^b = 0$.

9.4. Let T_{ab} be a tensor with $\nabla^a T_{ab} = 0$ and $T_{ab} = (\rho + p) u_a u_b - g_{ab} p$ where ρ and p are scalars, with $u_a u^a = 1$. Futhermore suppose that ρ is a function of pressure, and define

$$f = \exp \int_0^{p(x)} \frac{dp'}{p' + \rho(p')}$$

$$\hat{g}_{ab} = f^2 g_{ab}, \quad \hat{g}^{ab} = f^{-2} g^{ab}, \quad \hat{C}_a = f u_a, \quad \hat{c}^a = f^{-1} u^a.$$

Show that with respect to $\hat{g}_{ab}$ and the associated connection $\hat{\nabla}_a$, the current $\hat{C}_a$ is geodesic:

$$\hat{C}^a \hat{\nabla}_a \hat{C}^b = 0.$$

Solution: First we shall calculate

$$\partial_a(f^n) = \partial_a \exp \int_0^{p(x)} \frac{n\ dp'}{p' + \rho(p')} = \left(\exp \int_0^{p(x)} \frac{n\ dp'}{p' + \rho(p')} \right) \partial_a \left(\int_0^{p(x)} \frac{n\ dp'}{p' + \rho(p')} \right)$$

$$= \frac{n f^n \partial_a p}{p + \rho(p)}.$$

From this, we find that the connection is

$$\hat{\Gamma}^d_{bc} = \frac{1}{2} \hat{g}^{da} (\partial_b \hat{g}_{ca} + \partial_c \hat{g}_{ba} - \partial_a \hat{g}_{bc})$$

$$= \frac{1}{2} f^{-2} g^{da} (\partial_b (f^2 g_{ca}) + \partial_c (f^2 g_{ba}) - \partial_a (f^2 g_{bc}))$$

$$= \Gamma^d_{bc} + \frac{1}{2} f^{-2} \left(\frac{2\delta^d_c f^2 \partial_b p}{p + \rho(p)} + \frac{2\delta^d_b f^2 \partial_c p}{p + \rho(p)} - \frac{2 g_{bc} f^2 \partial^d p}{p + \rho(p)} \right)$$

$$= \Gamma^d_{bc} + \frac{\partial_a p}{p + \rho(p)} \left(\delta^d_c \delta^a_b + \delta^d_b \delta^a_c - g_{bc} g^{ad} \right).$$

Using the fact that $\nabla^a T_{ab}$ is zero, we find that

$$0 = \nabla^a T_{ab} = \nabla^a ((p + \rho) u_a u_b - g_{ab} p) = (p + \rho) u_b \nabla^a u_a + (p + \rho) u_a \nabla^a u_b + u_b u_a \nabla^a (p + \rho) - \nabla_b p$$

$$= (p + \rho) u_b \nabla^a u_a + (p + \rho) u_a \nabla^a u_b + u_b u_a \left(1 + \frac{d\rho}{dp} \right) \nabla^a p - \nabla_b p.$$

Contracting both sides by u^b, we find

$$(p + \rho) \nabla^a u_a + (p + \rho) u_a u^b \nabla^a u_b + \frac{d\rho}{dp} u_a \nabla^a p = 0.$$

But as shown in question 9.2, if $u^a u_a = 0$ then so does $u^b \nabla^a u_b$, and hence

$$(p+\rho)\nabla^a u_a + \frac{d\rho}{dp} u_a \nabla^a p = 0.$$

So we find that

$$\begin{aligned}
\nabla^a T_{ab} &= (p+\rho) u_b \nabla^a u_a + (p+\rho) u_a \nabla^a u_b + u_b u_a \left(1 + \frac{d\rho}{dp}\right) \nabla^a p - \nabla_b p \\
&= u_b \left((p+\rho)\nabla^a u_a + \frac{d\rho}{dp} u_a \nabla^a p\right) + u_a \nabla^a u_b + u_b u_a \nabla^a p - \nabla_b p \\
&= u_a \nabla^a u_b + u_b u_a \nabla^a p - \nabla_b p = 0.
\end{aligned}$$

Now we can calculate

$$\begin{aligned}
\hat{C}^a \hat{\nabla}_a \hat{C}^b &= f^{-1} u^a \partial_a (f^{-1} u^b) + f^{-1} u^a \hat{\Gamma}^b_{ac} (f^{-1} u^c) \\
= f^{-2} u^a \partial_a u^b + f^{-2} u^a \Gamma^b_{ac} u^c &- \frac{u^a u^b f^{-2} \partial_a p}{p+\rho(p)} + f^{-2} u^a \frac{\partial_d p}{p+\rho(p)} \left(\delta^d_c \delta^b_a + \delta^d_a \delta^b_c - g_{ac} g^{bd}\right) u^c \\
&= f^{-2} u^a \left(\nabla_a u^b - \frac{u^b \partial_a p}{p+\rho(p)} + \frac{\partial_d p}{p+\rho(p)} (\delta^b_a u^d + \delta^d_a u^b - g^{bd} u_a)\right) \\
&= f^{-2} \left(u^a \nabla_a u^b + \frac{u^b u^a \nabla_a p}{p+\rho(p)} - \frac{\nabla^b p}{p+\rho(p)}\right) \\
= \frac{f^{-2}}{p+\rho(p)} &\left((p+\rho(p)) u_a \nabla^a u^b + u^b u_a \nabla^a p - \nabla^b p\right) = \frac{f^{-2}}{p+\rho(p)} g^{bc} \nabla^a T_{ac} = 0,
\end{aligned}$$

which means that $\hat{C}^a$ is geodesic.

9.6. Suppose K_{ab} is a symmetric tensor satisfying

$$\nabla_{(a} K_{bc)} = \zeta_{(a} g_{bc)}$$

where ζ_a is a vector satisfying

$$\nabla_{(a} \zeta_{b)} = \frac{1}{2} k g_{ab}$$

for constant k. Show that

$$Q = k s^2 L^2 - 2sL\zeta_a u^a + K_{ab} u^a u^b$$

is constant along geodesic γ, where u^a is tangent to γ, s is the geodesic parameter and $L = \frac{1}{2} g_{ab} u^a u^b$.
Solution: Without loss of generality, we can assume that the geodesic has been parameterised so that $L = 1$. Now we can calculate

$$\begin{aligned}
\frac{dQ}{ds} &= 2ks - 2\zeta_a u^a - 2s \frac{D}{Ds}(\zeta_a u^a) + \frac{D}{Ds}(K_{ab} u^a u^b) \\
&= 2ks - 2\zeta_a u^a - 2s u^a u^b \nabla_b \zeta_a + u^a u^b u^c \nabla_c K_{ab}. \\
&= 2ks - 2\zeta_a u^a - 2s u^a u^b \nabla_{(b} \zeta_{a)} + u^a u^b u^c \nabla_{(c} K_{ab)} \\
&= 2ks - 2\zeta_a u^a - 2ks u^a u^b g_{ab} + u^a u^b u^c \zeta_{(c} g_{ab)} \\
&= 2ks - 2\zeta_a u^a - 2ks + 2u^c \zeta_c = 0
\end{aligned}$$

and hence Q is conserved along the geodesic.