Truncated Wigner Simulation of Squeezing in an Anharmonic Oscillator

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1 Introduction

As a quantum state evolves in a potential, the variances of observables will also change. If the variance decreases, then we say that the state has been squeezed. Squeezing is useful in a variety of applications, such as for reducing quantum noise and in the preparation of quantum states. We shall study the squeezing of a coherent state in the anharmonic oscillator governed by the Hamiltonian

$$\hat{H} = \omega \hat{a}^\dagger \hat{a} + u(\hat{a}^\dagger)^2 \hat{a}^2.$$

The Wigner function allows us to represent our system with a distribution $W(\alpha, \alpha^*)$ in phase space. In this representation, a coherent state centered on α_0 is given by the gaussian:

$$W(\alpha, \alpha^*) = \frac{2}{\pi} e^{-2|\alpha - \alpha_0|^2}.$$

Operators acting on a density matrix $\hat{\rho}$ can be represented by differential operators on $W(\alpha, \alpha^*)$

$$\hat{a}\hat{\rho} \rightarrow \left(\alpha + \frac{1}{2} \frac{\partial}{\partial \alpha^*} \right) W(\alpha, \alpha^*)$$

$$\hat{\rho}\hat{a} \rightarrow \left(\alpha - \frac{1}{2} \frac{\partial}{\partial \alpha^*} \right) W(\alpha, \alpha^*).$$

Note that transposing the left-hand side is equivalent to conjugating the right-hand side, providing us the representations of $\hat{a}^\dagger$ for free. These representations allows us to translate equations in terms of $\hat{\rho}$ into ones involving the Wigner function.

To study the anharmonic oscillator, we need the equation of motion for the Wigner function. If we represented the anharmonic oscillator using a density matrix, then the master equation would be

$$\frac{d\hat{\rho}}{dt} = -i[\hat{H}, \hat{\rho}]$$

where we have set $\hbar = 1$ for convenience. By expanding the right-hand side in terms of $\hat{a}$ and $\hat{a}^\dagger$ and then applying the operator correspondences, we get an equation for the Wigner function. For our anharmonic oscilllator, we find that the Wigner function evolves according to the equation

$$\frac{\partial W}{\partial t} = i \left[(\omega - 2u) \left(\frac{\partial}{\partial \alpha} \alpha - \frac{\partial}{\partial \alpha^*} \alpha^* \right) \right.$$

$$-2u \left(\frac{\partial}{\partial \alpha} \alpha |\alpha|^2 - \frac{\partial}{\partial \alpha^*} \alpha^* |\alpha|^2 \right)$$

$$\left. + \frac{u}{2} \left(\frac{\partial^3}{\partial \alpha^2 \partial \alpha^*} \alpha - \frac{\partial^3}{\partial (\alpha^*)^2 \partial \alpha} \alpha^* \right) \right] W.$$

We will study the squeezing of quadrature operators. Using the Wigner function, the quadratures can be calculated as

$$\hat{X}_\theta = \frac{1}{2} \langle e^{i\theta}\alpha + e^{-i\theta}\alpha \rangle = \langle \text{Re}(i\theta\alpha) \rangle,$$

where, as the Wigner function is a quasi-probability distribution, we denote

$$\langle f(\alpha, \alpha^*) \rangle = \int f(\alpha, \alpha^*) W(\alpha, \alpha^*) \, d\alpha.$$

The variance of quadratures is given by

$$\text{Var} X_\theta = \langle \text{Re}(e^{i\theta}\alpha)^2 \rangle - \langle \text{Re}(e^{i\theta}\alpha) \rangle^2.$$

We now use the expansion

$$\text{Re}(z) = \frac{1}{2}(z^* + z)$$

to rewrite the variance as

$$\text{Re}\left((\langle \alpha^2 \rangle - \langle \alpha \rangle^2) e^{2i\theta} \right) + \langle |\alpha|^2 \rangle - |\langle \alpha \rangle|^2.$$

By defining the variables

$$Z = \langle \alpha^2 \rangle - \langle \alpha \rangle^2$$

$$C = \langle |\alpha|^2 \rangle - \langle |\alpha| \rangle^2,$$

we formulate the expression

$$\text{Var}(\hat{X}_\theta) = \text{Re}(Ze^{2i\theta}) + C$$

for the variance. The numbers Z and C are the only values we need calculate in order to know the variance in any direction. The value of $\text{Re}(Ze^{2i\theta})$ varies between $-|Z|$ and $|Z|$, and so we have

$$\min(\text{Re}(Ze^{2i\theta}) + C) = C - |Z|.$$

In a coherent state, the variance of any quadrature is $\frac{1}{2}$, so squeezing occurs if

$$C - |Z| < \frac{1}{2}.$$

2 The Truncated Wigner Method

As partial differential equations are difficult to solve, we can study squeezing via the truncated Wigner method. In this approach, we ignore the third-order terms and simply try to solve

$$\frac{\partial W}{\partial t} = i\left[(\omega - 2u)\left(\frac{\partial}{\partial \alpha}\alpha - \frac{\partial}{\partial \alpha^*}\alpha^*\right)\right.$$

$$\left. -2u\left(\frac{\partial}{\partial \alpha}\alpha|\alpha|^2 - \frac{\partial}{\partial \alpha^*}\alpha^*|\alpha|^2\right)\right]W.$$

We use a Green's function style method to unravel this equation. We will assume the ansatz $W(\alpha, t) = \delta(\beta(t) - \alpha)$; this is the general form of a delta function centered on $\beta(t)$ in phase space. Substituting this into the PDE gives us

$$\delta'(\beta(t) - \alpha)\beta'(t)$$

$$= i(w - 2u)\beta(t)\delta'(\beta(t) - \alpha) - 2u\beta(t)|\beta(t)|^2\delta'(\beta(t) - \alpha),$$

Dividing through by $\delta'(\beta(t) - \alpha)$, we get the ODE

$$\frac{d\beta}{dt} = i[(\omega - 2u) - 2u|\beta|^2]\beta dt.$$

Because the PDE is linear, we can think of the Wigner function as being comprised of a infinite sum of these δ-functions. We hence conclude that the Wigner function evolves like an ensemble of particles following the equation

$$\frac{d\alpha}{dt} = i[(\omega - 2u) - 2u|\alpha|^2]\alpha dt$$

where we have substituted β for α for future convenience. Hence, we can simulate the Wigner function as the limit of an infinite number of particles evolving according to the above equation, with the distribution of the particles given by the initial Wigner function.

First note that $|\alpha|^2$ is a constant of motion:

$$\frac{d}{dt}|\alpha|^2 = \alpha^*\frac{d\alpha}{dt} + \alpha\frac{d\alpha^*}{dt}$$

$$= i[(\omega - 2u) - 2u|\alpha|^2]|\alpha|^2 + (i[(\omega - 2u) - 2u|\alpha|^2])^*|\alpha|^2$$

$$= i[(\omega - 2u) - 2u|\alpha|^2]|\alpha|^2 - i[(\omega - 2u) - 2u|\alpha|^2]|\alpha|^2 = 0.$$

For this reason, if we denote $\alpha_0 = \alpha(t)$, then the equation of motion becomes

$$\frac{d\alpha}{dt} = i[(\omega - 2u) - 2u|\alpha_0|^2]\alpha.$$

As a first-order linear differential equation, we know that this equation has the analytic solution

$$\alpha(t) = \alpha_0 e^{i[(\omega - 2u) - 2u|\alpha_0|^2]t}.$$

Because the variance of a distribution does not change upon rotations, we note that if we instead use the equation:

$$\alpha(t) = \alpha_0 e^{-i2u|\alpha_0|^2 t}.$$

then neither $|Z|$ nor C will change. This however, has the effect of rotating Z by an angle of $(\omega - 2u)t$; in fact, we can view the ignoring of the $(\omega - 2u)t$ term as being equivalent to entering a coordinate system rotating with angular velocity $\omega - 2u$. This also demonstrates that the value of ω does not effect the magnitude of squeezing that occurs.

We will use two different methods to calculate Z and C. Our first method is to sample the initial distribution, use our equation to calculate the location of the points at some future time, and then average over the observables $\langle\alpha\rangle$ and $\langle\alpha^2\rangle$. These two variables shall be sufficient to calculate Z and C. As $|\alpha|$ is a constant of motion, we will not need to calculate $\langle|\alpha|^2\rangle$ except for the initial distribution. Python code for this method is included in the appendix.

Instead of using a stochastic method to calculate the averages, we can analytically find our expectation values. This is only possible because we were able to analytically solve the differential equation for α; for more complicated systems we would have to content ourself with a stochastic method. We calculate expectation values using the formula:

$$\langle f(\alpha(t))\rangle = \int W(\alpha_0)f(M(\alpha_0, t))d\alpha_0$$

where $W(\alpha_0)$ is the initial Wigner function, and where M is the function:

$$M(\alpha_0, t) = \alpha_0 e^{-i2u|\alpha_0|^2 t}$$

which takes an initial point and evolves it forward by t. With the help of *Mathematica*, we find that:

$$\langle\alpha(t)\rangle = \frac{-r}{(ut - i)^2}e^{\frac{2utr^2}{i - ut}}$$

$$\langle\alpha(t)^2\rangle = \frac{r^2}{(1 + 2iut)^2}e^{\frac{4utr^2}{i - 2ut}}$$

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which allows us to find Z and C analytically.

As an example, we shall explore the system for $u = 0.0005$, beginning in the coherent state with $\alpha = 100$. We will simulate the system over the time period $t \in [0, 0.5]$. *Figure 1* shows the evolution of $|Z|$ and C over time computed analytically. In this graph, we can see that both these quantities rapidly increase, and have roughly the same value for most of the time interval. The minimum variance, given by $|Z| - C$, is plotted in *Figure 2*. On this graph, we can see that squeezing occurs, with the minimum minimum occurring at around $t = 0.275$ This minimum is very small, with a variance of about 0.022475, which is over a twenty-fold reduction in variance. From these two diagrams, we can deduce that the Wigner function is significantly squeezed in one direction, and spreads out considerably in the other quadratures.

We now compare the accuracy of the stochastic method with the analytic solution. In *Figure 3*, we plot the relative error of $|Z|$, C, and the minimum variance, in the stochastic method. For the stochastic method, 10000000 trials were used. There is an initial spike in the relative error in $|Z|$, this is an artefact of the caused by the initial null value of $|Z|$. In this diagram we can see that the relative error is always below 0.15%, and if we ignore the initial spike in $|Z|$, is always below 0.04% for both $|Z|$ and C. The relative error for the minimum variance, $C - |Z|$, is larger, but is still always below 0.3%. From this we can conclude that the stochastic method provides a good approximation to the analytic solutions.

3   Direct Integration

In this section we shall attempt to directly integrate the equation:

$$\frac{\partial W}{\partial t} = i \left[(\omega - 2u) \left(\frac{\partial}{\partial \alpha} \alpha - \frac{\partial}{\partial \alpha^*} \alpha^* \right) \right.$$

$$- 2u \left(\frac{\partial}{\partial \alpha} \alpha |\alpha|^2 - \frac{\partial}{\partial \alpha^*} \alpha^* |\alpha|^2 \right)$$

$$\left. + \frac{u}{2} \left(\frac{\partial^3}{\partial \alpha^2 \partial \alpha^*} \alpha - \frac{\partial^3}{\partial (\alpha^*)^2 \partial \alpha} \alpha^* \right) \right] W$$

using a grid method. This will allow us to test the validity of the truncated Wigner method.

To have any hope of simulating the Wigner method in this way, we need to be able to choose a grid fine enough to simulate the region of phase-space where the Wigner function is large accurately, without being too slow. We cannot possible have a grid which allows simulation of the rotation of the Wigner function, so we need to try to use a grid which is centered on the Wigner function. To achieve this, we can choose a coordinate system so that the Wigner function fails to move on average. The term

with coefficient $\omega - 2u$ simple causes the Wigner function to rotate without modify its shape; for this reason, we can simply ignore it. Next we note that, from our truncated Wigner method, the second term causes a point in the Wigner function to follow the equation:

$$\alpha(t) = e^{-2ui|\alpha|^2 t}$$

and we can see that this would cause the entire Wigner function to rotate. However, it will also distort the Wigner function, so we cannot ignore this term. Instead, we can use a rotating frame which keeps the centre of the Wigner function at the same point; this corresponds to choosing a coordinate system:

$$\alpha' = \alpha e^{2ui|\alpha_0|^2 t}.$$

We can hence use the equation

$$\frac{\partial W}{\partial t} = i \left[2u \left(\frac{\partial}{\partial \alpha^*} \alpha^* (|\alpha|^2 - |\alpha_0|^2) - \frac{\partial}{\partial \alpha} \alpha (|\alpha|^2 - |\alpha_0|^2) \right) \right.$$

$$\left. + \frac{u}{2} \left(\frac{\partial^3}{\partial \alpha^2 \partial \alpha^*} \alpha - \frac{\partial^3}{\partial (\alpha^*)^2 \partial \alpha} \alpha^* \right) \right] W$$

to describe the evolution of the Wigner function. However, the first term in this equation is numerically unstable when

$$|\alpha|^2 - |\alpha_0|^2 < 0,$$

For this reason, rather than using $|\alpha_0|^2$, we use the equation

$$\frac{\partial W}{\partial t} = i \left[2u \left(\frac{\partial}{\partial \alpha^*} \alpha^* (|\alpha|^2 - |\alpha_0|^2 + \varepsilon) - \frac{\partial}{\partial \alpha} \alpha (|\alpha|^2 - |\alpha_0|^2 + \varepsilon) \right) \right.$$

$$\left. + \frac{u}{2} \left(\frac{\partial^3}{\partial \alpha^2 \partial \alpha^*} \alpha - \frac{\partial^3}{\partial (\alpha^*)^2 \partial \alpha} \alpha^* \right) \right] W$$

where ε is a small positive number. Eventually the Wigner function will spread out too much, and our grid method will break.

We shall again simulate the system for the case of $u = 0.0005$ beginning in the coherent state with $\alpha_0 = 100$. We will attempt to use a grid covering the points between $(90, -10)$ and $(110, 10)$. The results are plotted in Figure 4. Note that this plot only includes up to $t = 0.3$; we can see that the method has broken by $t = 0.25$, as can be confirmed by the fact that the minimum variance predicted is negative. In Figure 5, we plot the relative difference between the direct Wigner method and the analytic values calculated before for $|Z|$ and C. This is only plotted until $t = 0.21$, as beyond that point the difference grows rapidly. In this we can see that the percentage difference is consistenly below 0.01%. It is difficult to tell whether this difference is a result of numerical errors from the integration of the Wigner evolution, or whether it is

a result of the truncation of the Wigner function. However, the relative error is quite jagged, so this suggests that the former mechanism is at work. Regardless, the error caused by truncating the Wigner function is clearly very small in this parameter regime.

In principle we could use a larger grid in order to simulate squeezing for longer. However, our direct integration is not very robust, and so increasing the grid size, even whilst keeping the mesh size constant, does not necessarily improve the accuracy of the method. Furthermore, while naively one might expect that the time taken to be proportional to the number of elements in the grid, in practise the time taken increases more rapidly, as a smaller step size is required to keep the numerical errors in check. These are all issues which do not occurr in the truncated Wigner approximation.

4   The Quantum Regime

So far we have examined the case of $\alpha_0 = 100$ and $u = 0.0005$. For these parameters, quantum corrections to classical behaviour are small, and hence the truncated Wigner method is quite good. In this section, we shall explore the case where $\alpha_0 = 2$ and $u = 0.5$, a regime where quantum effects become important. We shall calculate $|Z|$ and C both analytically and also through direct integration. Fortunately, in this regime we are able to directly integrate the Wigner function without the method breaking down. These are plotted in *Figure 6*, for t between 0 and 0.5.

In *Figure 6*, we can see that the value of C increases over time, but the value of $|Z|$, while initial increasing, eventually reaches a maximum at around 0.37 before decreasing again. The analytic method is generally fairly close to the value calculated through direct integration, though the difference between the two method does increase over time. This shows us that the truncated third order terms are becoming increasingly relevant in this regime. If we were to simulate the system for a long time period, the truncated Wigner function would continue to diverge from the correct values until it became completely erroneous.

In *Figure 7*, the minimum quadrature variance calculate using both method is plotted. In this graph, we can see that squeezing occurrs, with the minimum variance being 0.258 at time $t = 0.115$. But we can see that squeezing stops at $t = 0.25$, and the minimum variance continues to increase past this time. The analytic method consistenly overestimates the minimum variance, and slowly but steadily diverges from the directly integrated value.

5   Conclusion

We have studied the squeezing of a coherent state in an anharmonic oscillator using the Wigner representation of the system. In order to evolve the Wigner function, we made use of the truncated Wigner method. This allowed us to treat the system as an ensemble of classical particles following fixed trajectories, which we then analytically solved. We explored the system in the semiclassical regime where the energy was large, and found that stochastically calculating expectation values was efficient and accurate. Whilst for this system we were able to calculate expectation values analytically, general this is impossible and so it is necesary to use such stochastic methods.

We were able to directly integrate the full equation of motion for the Wigner function. This allowed us to test the validity of the truncated Wigner function. For the semiclassical regime, the error in truncation was negligable. However, when we examined initial conditions which were close to the vacuum, we found that the truncating the third-order terms caused small but persistently increasing errors. On longer time scales, these errors would have eventually prevented the Wigner function from providing any accurate predictions about the actual system.

We found that squeezing occurred in both parameter regimes, though for our first simulation we saw far smaller minimum squeezing. In both the parameter regimes explored, the minimum variance initial decreased, before reaching a minimum and then growing again. In our second simulation, we saw that the squeezing only occurred temporarily, and that over longer periods of time the minimum variance grew larger.
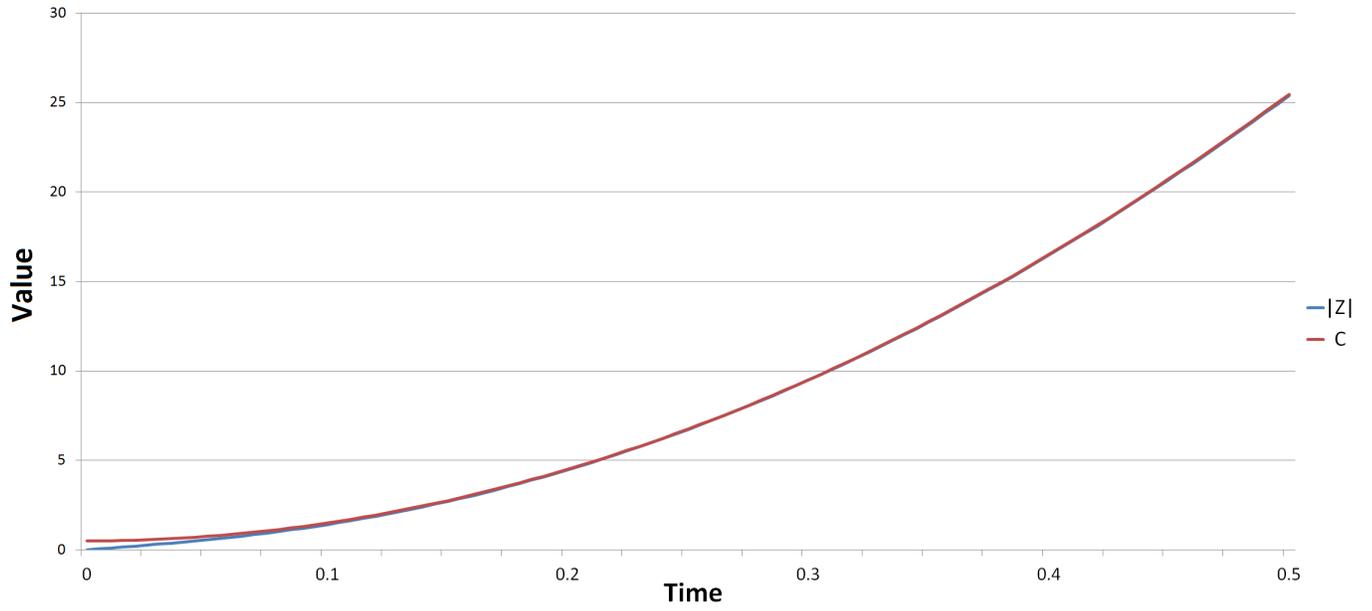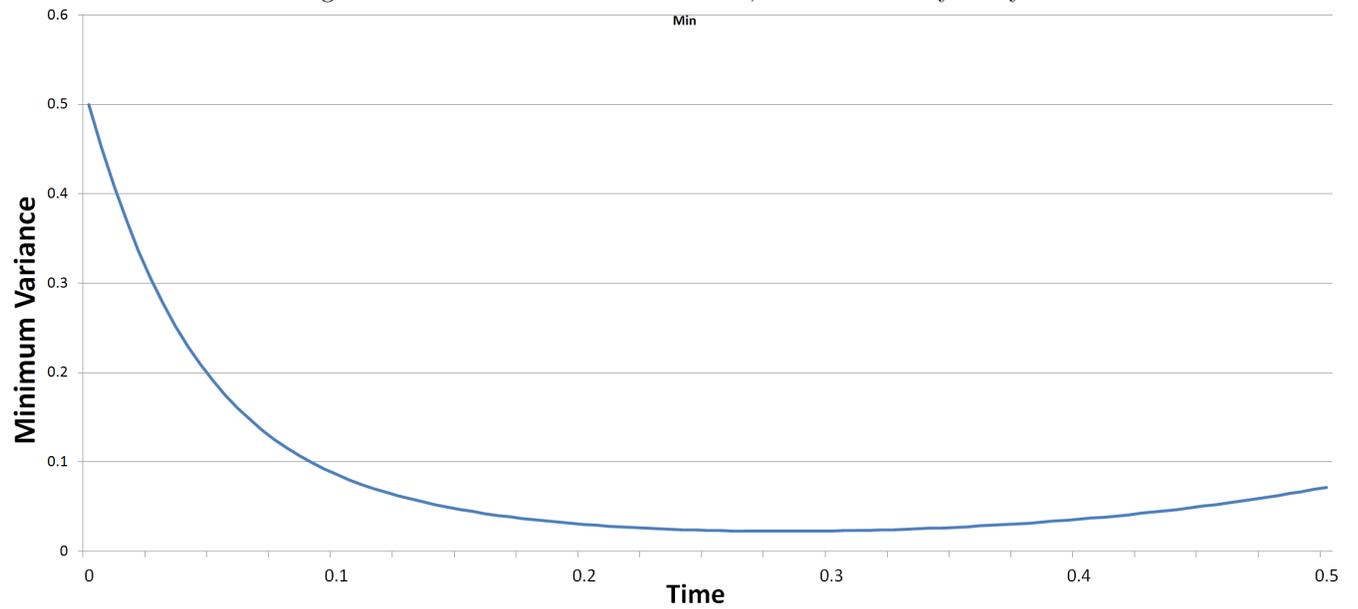
Figure 1: Evolution of $|Z|$ and C over time calculated analytically.

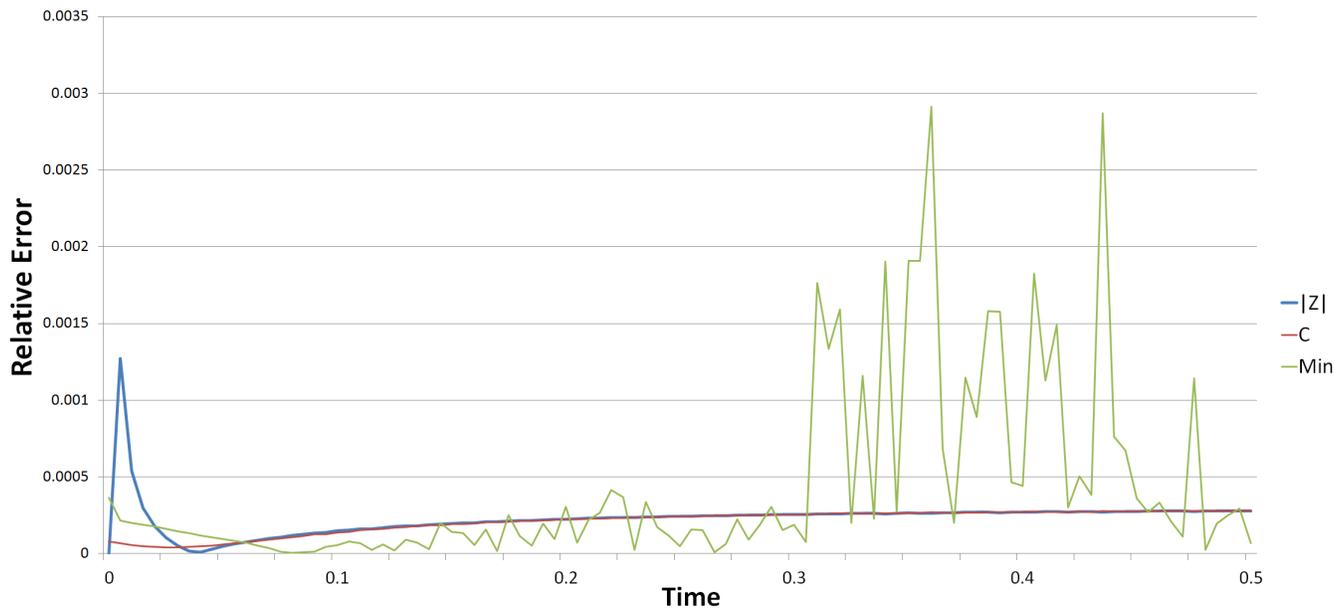Figure 2: Minimum Variance over time, calculated analytically.

Figure 3: Relative Error in the Stochastic Method for $|Z|$, C, and for the minimum uncertainty $C - |Z|$.
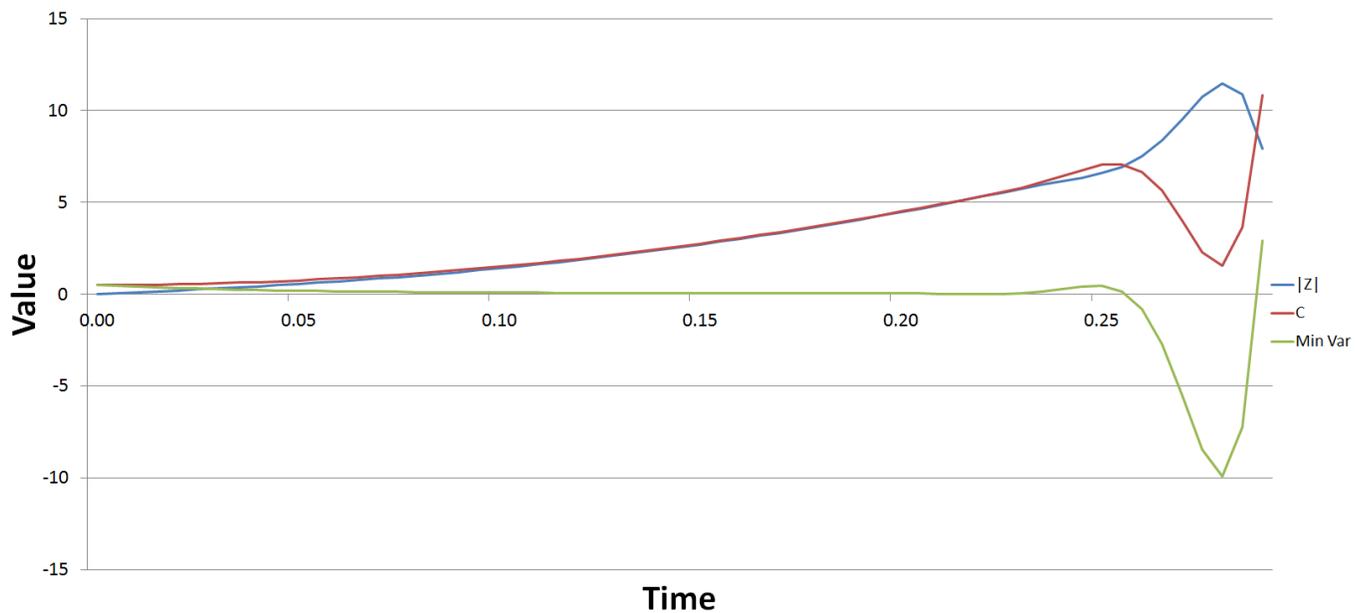

Figure 4: Value of $|Z|$, C, and the minimum variance $|Z| - C$, predicted using the direct integration method.
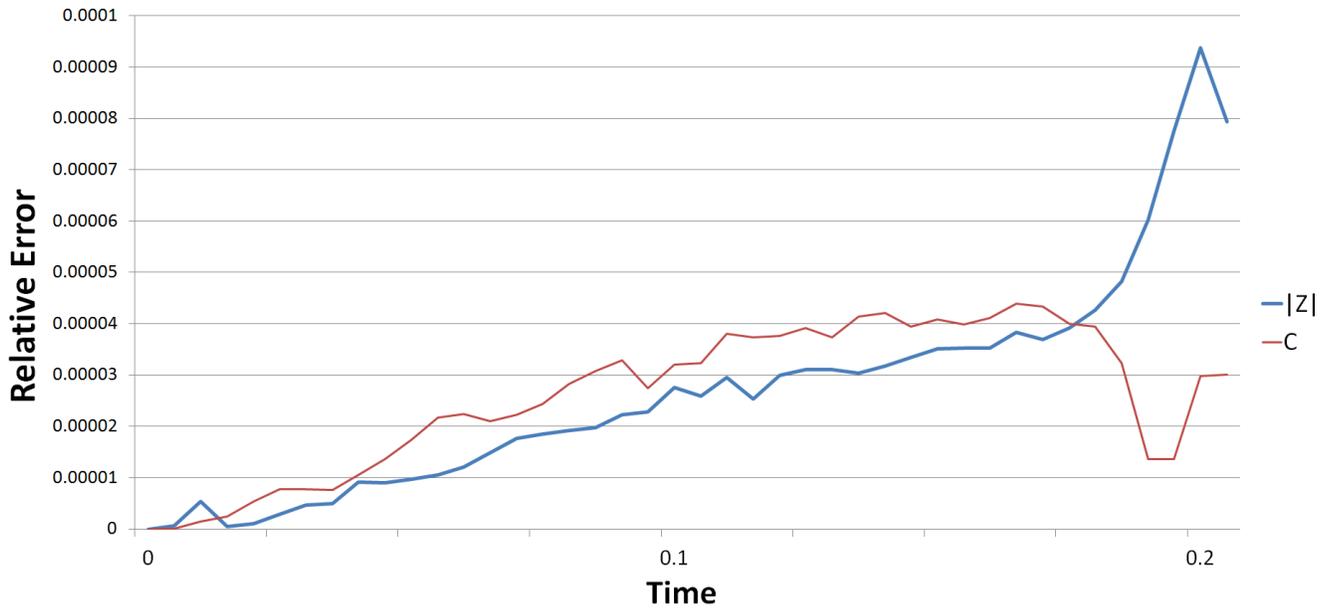
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Figure 5: Relative difference between the analytic solution and the direct integration of the Wigner function for $|Z|$ and C.
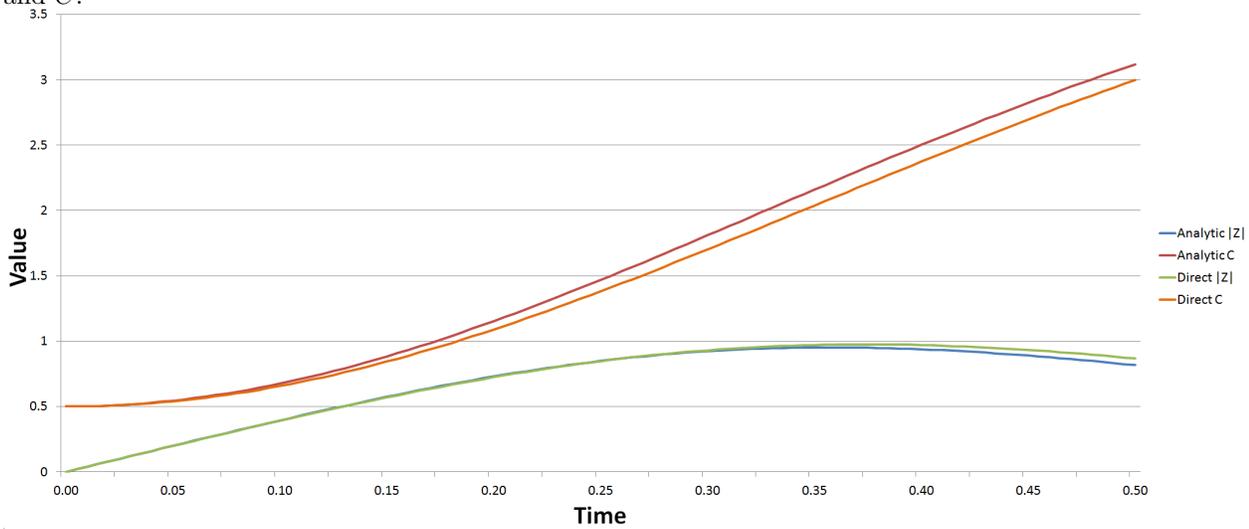

Figure 6: Values of $|Z|$ and C calculated both analytically and by directly integrating the Wigner function, for $u = 0.5$ and $\alpha_0 = 2$.
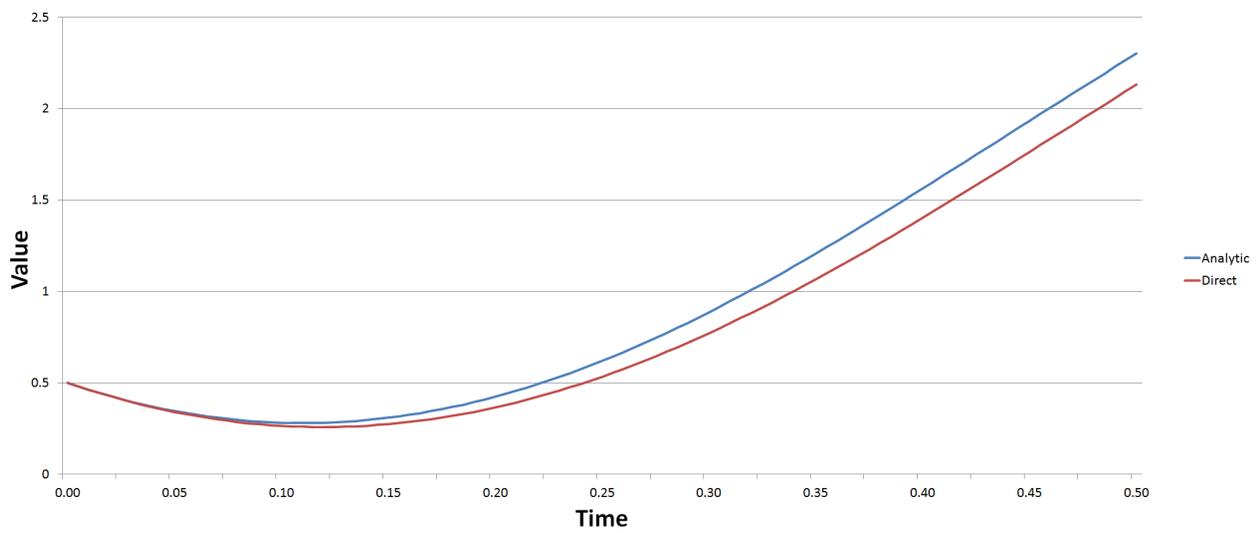
Figure 7: Minimum quadrature variance calculated analytically and through directly integrating the Wigner function.

6 Appendix: Code used for the Project

Python Code used to stochastically calculate expectation value.

```python
import cmath
import random

paths = 10000
u = 0.0005
r = 100

samples = 101
step = 0.005
u_step = 2*step*u*1j

AvA  = [0 for i in range(0,samples)]
AvAA = [0 for i in range(0,samples)]
AvAbsA= [0 for i in range(0,samples)]

for i in range(0,paths):
    a0 = random.gauss(r,1/2)+1j*random.gauss(0,1/2)
    r0 = abs(a0)**2
    for j in range(0,samples):
        a  = a0*cmath.exp(-u_step*r0*j)
        AvA[j]+= a
        AvAA[j]+=a**2
        AvAbsA[j]+=abs(a)**2
    if i%10000==0:
        print(i)

for i in range(0,samples):
    Z = AvAA[i]/paths-AvA[i]**2/paths**2
    C = AvAbsA[i]/paths-abs(AvA[i])**2/paths**2
    print(Z.real," ",Z.imag," ",C)
```

This is the XMDS code used to directly integrate the Wigner function.

```xml
<?xml version="1.0" encoding="UTF-8"?>
    <simulation xmds-version="2">

    <name>Direct</name>
    <author>Damon Binder; based on code by Graham Dennis and Joe Hope</author>
    <features>
        <benchmark />
        <bing />
        <fftw plan="patient" />
        <globals>
            <![CDATA[
            /* system constants */
            const real omega = 0;
            const real u = 0.001;
            /* initial state constants */
            const real alpha_0 = 100;
            /* used for calculating constants derived from arguments */
```

```
        bool firstTime = true;
        /* absorbing boundary constants */
        const real absorb = 8.0;
        const real width = 0.3;
        /* derived constants */
        const complex miOmega = -i*omega;
        const complex miUint_hbar = -i*u;
        const real Uint_hbar = u;
        ]]>
        </globals>
</features>

<geometry>
    <propagation_dimension> t </propagation_dimension>
    <transverse_dimensions>
    <dimension name="x" lattice="128" domain="(80, 110)" />
    <dimension name="y" lattice="128" domain="(-10, 10)" />
    </transverse_dimensions>
</geometry>

<vector name="main" initial_basis="x y" type="complex">
    <components> W </components>
    <initialisation>
        <![CDATA[
        W = 2.0/M_PI * exp(-2.0*(y*y + (x-alpha_0)*(x-alpha_0)));
        ]]>
    </initialisation>
</vector>

<vector name="dampConstants" initial_basis="x y" type="real">
    <components>damping</components>
    <initialisation>
        <![CDATA[
        real r = sqrt(x*x + y*y);
        if (r > _max_x-width)
        damping = 0.0;
        else
        damping = 1.0;
        ]]>
    </initialisation>
</vector>

<sequence>
    <integrate algorithm="ARK89" tolerance="1e-7" interval="0.5" steps="100">
    <samples>100</samples>
    <operators>
        <operator kind="ex">
        <operator_names>Lx Ly Lxxx Lxxy Lxyy Lyyy</operator_names>
        <![CDATA[
        Lx = i*kx;
        Ly = i*ky;
        Lxxx = -i*kx*kx*kx;
        Lxxy = -i*kx*kx*ky;
```

```
            Lxyy = -i*kx*ky*ky;
            Lyyy = -i*ky*ky*ky;
            ]]>
        </operator>

        <integration_vectors>main</integration_vectors>
            <dependencies>dampConstants</dependencies>
            <![CDATA[
            dW_dt = damping * (
            (Uint_hbar*(x*x + y*y+5-alpha_0*alpha_0))*x*Ly[W]
            - (Uint_hbar*(x*x + y*y+5-alpha_0*alpha_0))*y*Lx[W]
            - Uint_hbar/16.0*(x*(Lxxy[W] + Lyyy[W])
            -y*(Lxyy[W] + Lxxx[W]))
            );
            ]]>
            </operators>
        </integrate>
    </sequence>

    <output filename="Direct.xsil" format="ascii">
        <sampling_group basis="x(0) y(0)" initial_sample="yes">
        <moments>aR aI aaR aaI aC</moments>
        <dependencies>main</dependencies>
            <![CDATA[
            aR = x*W.Re();
            aI = y*W.Re();
            aaR= (x*x-y*y)*W.Re();
            aaI= (2*x*y)*W.Re();
            aC = (x*x+y*y)*W.Re();
            ]]>
        </sampling_group>
    </output>
</simulation>
```