Supersymmetric Quantum Mechanics

Damon Binder
PHYS3001 Project 2

September 29, 2014

Abstract

In this paper, the application of supersymmetry to the solving of quantum mechanical Hamiltonians is explored. A method for factorising Hamiltonians is developed and it is shown that factorisation allows one to construct a second Hamiltonian with an almost identical eigenspectra. This is then applied to solving Shape Invariant Potentials (SIPs), and the close relationship between SIPs and analytically solvable potentials is discussed.

1 Introduction

One of the major goals of modern physics has been to unify the various symmetries found in nature. Initially symmetries which were generated by a Lie algebra were considered; these are sets of symmetries which are closed only under commutation. However, a no-go theorem discovered by Coleman and Mandula in 1967[1] proved that it would not be possible to achieve this unification through Lie algebras. In order to circumvent the theorem, superalgebras were developed, which, unlike Lie algebras, are closed under both commutation and anticommutation. The type of symmetries generated by these superalgebras was known as supersymmetries (SUSY).

In order to study supersymmetric quantum field theories, supersymmetric quatum mechanics (SUSY QM) was developed. This was initial used as a toy model in order to test mathematical methods that could be applied to the more complicated field theories. In Section 2 we begin by introducing one such model, originally studied by Witten in [2]. It was soon discovered that these models were of interest in their own right, as they could be used to create Hamiltonians pairs with closely related eigenspectras.

There is a close relationship between SUSY QM and Hamiltonian factorisation, which we shall explore in Section 3. The latter method was explored by Infeld and Hull [3] in the 1950s; however, the inimate connection between factorisation and supersymmetry would only be discovered thirty years later. The results derived in Section 3 are applied to scattering states in Section 4.

In Section 5, repeated factorisation is used to create a hierachy of Hamiltonians from a given initial Hamiltonian, and this leads to the concept of shape invariant potentials (SIP) in Section 6, a concept first introduced by Gendenshtein in 1983 [4]. For this class of potentials, the energy eigenvalues can easily be calculated using algebraic methods. Curiously, all standard solvable potentials, such as the harmonic oscillator, the free particle, and the infinite square well, are shape invariant potentials, demonstrating an intimate connection between analytically solvable problems in QM, supersymmetry, and the factorisation method.

2 Supersymmetric QM

Historically, SUSY QM arose out of the attempts of physicists to create toy models of SUSY which they could use as a testing ground for the theory. We shall take a spin 1/2 model where:

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}$$

is the Hamiltonian, and where:

$$Q = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}$$

is known as the supercharge operator. These shall satisfy the anti-commutator relations:

$$\{Q, Q\} = 0, \quad \{Q, Q^T\} = H \tag{1}$$

This algebra was first introduced by Nicolai[5]. The first anticommutator relationship is equivalent to stating that Q is nilpotent. By expanding the second anticommutator relation, we find:

$$H = QQ^T + Q^TQ = \begin{pmatrix} A^TA & 0 \\ 0 & AA^T \end{pmatrix}$$

While SUSY QM originated in order to understand the properties of H, it was soon found that H_1 and H_2 were closely related, and that this could be used to solve one-dimensional problems in QM. To see this, let:

$$H_1|\psi_n^1\rangle = E_n^1|\psi_n^1\rangle$$

be the eigenvectors of H_1. For $n > 0$:

$$H_2 A|\psi_n^1\rangle = AA^T A|\psi_n^1\rangle = AH_1|\psi_n^1\rangle = E_n^1 A|\psi_n^1\rangle \quad (2)$$

Likewise, if $H_2|\psi_n^2\rangle = E_n^2|\psi_n^2\rangle$, then:

$$H_1 A^T|\psi_n^2\rangle = A^T H_2|\psi_n^2\rangle = E_n^2 A^T|\psi_n^2\rangle \quad (3)$$

These relationships tell us that, except for states of zero energy, the two Hamiltonians have an identical energy spectra, and closely related eigenvectors. These degeneracies are in fact a result of the supersymmetry of H:

$$[H, Q] = HQ - QH$$
$$= QQ^TQ + Q^TQ^2 - Q^2Q^T - QQ^TQ = 0$$

In other words, as H and Q commute, Q can be seen as a symmetry of H, and this symmetry results in the degenarate eigenvalues of H.

3 Factorisation of a Hamiltonian

In the previous section we found that if we have Hamiltonians of the form A^TA and AA^T, then they have an (almost) identical spectra and closely related eigenvectors as a result of supersymmetry. In order to apply our results to quantum mechanical problems, we need a method of factorising Hamiltonians into the form A^TA. Our treatment of factorisation loosely follows the treatment in [6].

Let us begin with the Hamiltonian of a single particle:

$$H_1 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_1(x)$$

which has a ground state of zero energy. This requirement does not restrict us, as we can always add a constant to the Hamiltonian in order to achieve this.

We wish to factorise the Hamiltonian as $H_1 = A^TA$. In order to do so we can use the ansatz:

$$A = \frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x), \quad A^T = -\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x)$$

where $W(x)$ is known as the superpotential [6]. Then:

$$H_1 = \left(-\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x)\right)\left(\frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x)\right)$$
$$= \frac{\hbar^2}{2m}\frac{d^2}{dx^2} + W(x)^2 - \frac{\hbar}{\sqrt{2m}}W'(x)$$

Let ψ_0 be the ground state of H_1. Then:

$$\langle\psi_0|H_1|\psi_0\rangle = \langle A\psi_0|A\psi_0\rangle = 0$$

This implies that $A|\psi_0\rangle = 0$, and thus:

$$\frac{\hbar}{\sqrt{2m}}\frac{d\psi_0}{dx} + W(x)\psi_0(x) = 0$$

We can then rearrange this to find the superpotential:

$$W(x) = -\frac{\hbar}{\sqrt{2m}}\frac{\psi_0'(x)}{\psi_0(x)} = -\frac{\hbar}{\sqrt{2m}}\frac{d\ln\psi_0}{dx} \quad (4)$$

We can now calculate the SUSY partner Hamiltonian:

$$H_2 = AA^T = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + W(x)^2 - \frac{\hbar}{\sqrt{2m}}W'(x)$$

The potentials:

$$V_{1,2}(x) = W(x)^2 \pm \frac{\hbar}{\sqrt{2m}}W'(x) \quad (5)$$

are known as SUSY partner potentials. From (2) and (3) we know that the nonzero spectra of H_1 and H_2 are identical. We can, however, derive an even stronger relationship between H_1 and H_2 by proving that H_2 has no zero eigenvalues.

Assume that this is false, and that $H_2\psi_0^2(x) = 0$. Then:

$$0 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi_0^2(x) + \left(W(x)^2 - \frac{\hbar^2}{2m}W'(x)\right)\psi_0^2$$

which, by rearranging and applying (4), simplifies to:

$$-\frac{\hbar}{\sqrt{2m}}\frac{d}{dx}\ln\psi_0^1(x) = \frac{\hbar}{\sqrt{2m}}\frac{d}{dx}\ln\psi_0^2(x)$$

Therefore:

$$\psi_0^2(x) = C(\psi_0^1(x))^{-1}$$

which is not normalisable. Thus, H_2 has no zero eigenvalues.

As $E_0^1 = 0$ and $E_0^2 \neq 0$, we can use (2) and (3) to deduce that:

$$E_0^1 = 0, \quad E_n^2 = E_{n+1}^1 \quad (6)$$
$$|\psi_{n-1}^2\rangle = (E_n^1)^{-1/2}A|\psi_n^1\rangle \quad (7)$$
$$|\psi_{n+1}^1\rangle = (E_n^2)^{-1/2}A^T|\psi_n^2\rangle \quad (8)$$

where the normalisation constants in (7) and (8) can be found using:

$$\langle A\psi_n^1|A\psi_n^1\rangle = \langle\psi_n^1|A^TA|\psi_n^1\rangle = \langle\psi_n^1|H_1|\psi_n^1\rangle = E_n^1$$
$$\langle A\psi_n^2|A\psi_n^2\rangle = \langle\psi_n^2|AA^T|\psi_n^2\rangle = \langle\psi_n^2|H_2|\psi_n^2\rangle = E_n^2$$

As an example, we will find the partner potential of the infinite square well. Modifying the potential in the centre so that the ground energy is 0, we find that:

$$E_n^1 = \frac{(n+1)^2\pi^2\hbar^2}{2mL^2} - \frac{\pi^2\hbar^2}{2mL^2} = \frac{n(n+2)\pi^2\hbar^2}{2mL^2}$$

$$\psi_n^1(x) = \left(\frac{2}{L}\right)^{1/2} \sin\left(\frac{(n+1)\pi x}{L}\right)$$

The superpotential of the infinite square well is [6]:

$$W(x) = -\frac{\hbar\pi}{L\sqrt{2m}}\frac{\cos(\pi x/L)}{\sin(\pi x/L)} = -\frac{\hbar\pi}{L\sqrt{2m}}\cot\left(\frac{\pi x}{L}\right)$$

and the partner potential is:

$$V_2(x) = \frac{\hbar^2\pi^2}{2mL^2}\left(\csc^2\left(\frac{\pi x}{L}\right) - 1\right)$$

For both $W(x)$ and $V_2(x)$, the functions are defined on the interval $[-L, L]$, and are infinite otherwise.

By shifting $V_2(x)$ by $\frac{\hbar^2\pi^2}{2mL^2}$, we can deduce the eigenvalues of:

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{\hbar^2\pi^2}{2mL^2}\csc^2\left(\frac{\pi x}{L}\right)$$

are the same as those of the infinite square well excluding the ground state, and that the eigenvectors can be found by multiplying the eigenvectors of the infinite square well by A. Thus, for H, the first two eigenvalues and eigenstates are:

$$E_0 = \frac{4\hbar^2\pi^2}{2mL^2}, \quad \psi_0(x) = -2\sqrt{\frac{2}{3L}}\sin^2\left(\frac{\pi x}{L}\right)$$

$$E_1 = \frac{9\hbar^2\pi^2}{2mL^2}, \quad \psi_1(x) = -\frac{2}{\sqrt{L}}\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi x}{L}\right)$$

4 SUSY QM and Scattering States

We shall now consider the applications of supersymmetric quantum mechanics to Hamiltonians with scattering states. In order for scattering to occur for partner potentials V_1 and V_2, it is necessary that these potentials have a finite limit as $x \to \pm\infty$. Let W be the SUSY potential. It follows from (5) that if $\lim_{x\to\pm\infty} W(x) = W_\pm$, then:

$$\lim_{x\to\pm\infty} V_{1,2} = W_\pm^2$$

Let $\psi_E^{1,2}(x)$ be a scattering eigenstate of energy E of $V_{1,2}$. For $V_{1,2}$ there is a transmission coefficient $T_{1,2}(E)$ and a reflection coefficient $R_{1,2}(E)$ such that:

$$\lim_{x\to\infty} \psi_E^{1,2}(x) = T_{1,2}e^{ik_+x}$$

$$\lim_{x\to-\infty} \psi_E^{1,2}(x) = e^{ikx} + R_{1,2}e^{-ik_-x}$$

where:

$$k_\pm = (E \pm W_-^2)^{1/2}$$

We can now use relationships (2) and (3) to derive:

$$\lim_{x\to\pm\infty} \psi_E^2(x) = \lim_{x\to\pm\infty} NA\psi_E^1(x)$$

where N is an arbitary normalisation constant. Thus:

$$T_1e^{ik_+x} = NT_2(-ik_+ + W_+)E^{ik_+x}$$

$$e^{ik_-x} + R_1e^{-ik_-x} = N[(-ik_- + W_-)e^{ik_-x} + (ik_- + W_-)e^{-k_-x}R_2]$$

Equating the coefficients of the exponential terms:

$$R_1(E) = \frac{W_- + ik_-}{W_- - ik_-}R_2(E)$$

$$T_1(E) = \frac{W_+ - ik_+}{W_- - ik_-}T_2(E) \tag{9}$$

From these relationships we can deduce $|R_1|^2 = |R_2|^2$ and $|T_1|^2 = |T_2|^2$; in other words, the reflection and transmission probabilities of parnter potentials are identical. In the special case of $W_+ = W_-$, we have $T_1(E) = T_2(E)$, and thus in this special case the phase shift of transmission are also equal.

From the previous discussion, we can conclude that a potential is reflectionless, then so is its partner potential. As an example, let us consider the superpotential $W(x) = -A\tanh(\alpha x)$. The partner potentials for this superporential are:

$$V_{1,2} = A^2 + A\left(A \pm \alpha\frac{\hbar}{2m}\right)\text{sech}^2(\alpha x)$$

When $A = \alpha\frac{\hbar}{2m}$, V_2 corresponds to a free particle and is therefore reflectionless. Thus:

$$V_1(x) = \alpha^2\frac{\hbar^2}{2m}(1 + 2\,\text{sech}^2(\alpha x))$$

is also reflectionless.

5 The Hierachy of Hamiltonians

In Section 3 we developed a method for factorising a Hamiltonian H_1 and then using this to create a SUSY partner H_2. There is nothing to stop us repeating this process, allowing us to generate a hierachy of Hamiltonians H_n, all of which can be solved if H_1 can be.

Let H_k have a ground state $\psi_0^k(x)$ with energy E_0^k. From the last section, we know that we can factorise:

$$H_k = A_k^T A_k + E_0^k = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_k(x)$$

where:

$$A_k = \frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W_k(x), \quad W_k(x) = -\frac{\hbar}{\sqrt{2m}}\frac{d}{dx}\ln\psi_0^k$$

We shall define $H_{k+1} = A_k A_k^T + E_0^k$, with the SUSY partner potential given by:

$$V_{k+1}(x) = W_k(x)^2 - \frac{\hbar}{\sqrt{2m}}W_k'(x) + E_0^k$$

$$= V_k(x) - \frac{\hbar^2}{m}\frac{d^2}{dx^2}\ln\psi_0^k \tag{10}$$

This simplifies down to (5) when $E_0^k = 0$. All we have done is shifted the eigenspectra of both H_k and H_{k+1} by E_0^{k+1}, and thus can generalise (6) and (7) to:

$$E_n^{k+1} = E_{n+1}^k, \quad |\psi_{n-1}^{k+1}\rangle = (E_n^k - E_0^k)^{-1/2}A_k|\psi_n^k\rangle \tag{11}$$

Beginning with $k = 1$ and recursively applying (10) and (11), we find that, in terms of the eigenstates of the H_1:

$$E_n^k = E_{n+k-1}^1 \tag{12}$$

$$\psi_n^k = \left(\prod_{i=1}^{m-1}\frac{A_i}{\sqrt{E_{n+m-1}^1 - E_{i-1}^1}}\right)\psi_{n+m-1}^1 \tag{13}$$

$$V_k(x) = V_1(x) - \frac{\hbar}{m}\frac{d}{dx}\ln(\psi_0^1\psi_1^1...\psi_k^1) \tag{14}$$

From these relationships, it can be seen that if the eigenstates and eigenvalues of H_1 are known, then we can easily calculate the these for all other Hamiltonians in hierachy. Conversely, if we know all the potentials in the hierachy, then we can solve H_1 using (14).

6 Shape Invariant Potentials

Let $W(x, a_1)$ be a superpotential which depends on a set of parameters a_1. We say that this superpotential is shape invariant if:

$$W(x, a_i)^2 - \frac{\hbar}{\sqrt{2m}}W'(x, a_i)$$

$$= W(x, a_{i+1})^2 + \frac{\hbar}{\sqrt{2m}}W'(x, a_{i+1}) + R(a_i) \tag{15}$$

where $R(a_i)$ is a constant independent of x and where $a_{i+1} = f(a_i)$ for some function f [8]. Let:

$$V_{1,2}(x, a_1) = W(x, a_1)^2 \pm \frac{\hbar}{\sqrt{2m}}$$

Such a potential is known as a Shape Invariant Potential (SIP). From (4), we can deduce that:

$$E_0^1 = 0, \quad \psi_0^1(x, a_1) = N\exp\left(-\int W_1(y, a_1)\ dy\right) \tag{16}$$

where we shall assume that $W(x, a_1)$ is such that $\psi_0^1(x, a_1)$ is normalisable for all choices of a_1, and N is a normalisation constant.[1]

We can use (15) to deduce that:

$$V_2(x, a_1) = W(x, a_1)^2 - \frac{\hbar}{\sqrt{2m}}W'(x, a_1)$$

$$= W(x, a_2)^2 + \frac{\hbar}{\sqrt{2m}}W'(x, a_2) + R(a_1)$$

Continuing in this manner, we can construct a hierachy of Hamiltonians with potentials:

$$V_k = W(x, a_k)^2 + \frac{\hbar}{\sqrt{2m}}W'(x, a_k) + \sum_{j=0}^{k} R(a_j)$$

As the ground state of V_1 is zero, we can conclude that:

$$E_0^k = \sum_{j=1}^{k-1} R(a_j) \tag{17}$$

and using (12), can deduce the eigenspectra of H_1:

$$E_n^1 = \sum_{j=1}^{k-1} R(a_j) \tag{18}$$

Similarly, as we know the ground state of $V_1(x, a_1)$, we can deduce the eigenfunction of V_1. This can be seen by starting with H_k which has ground state $\psi_0^1(x, a_k)$, and then using (13):

$$A(x, a_j) = \frac{\hbar}{\sqrt{2m}}\frac{d}{dx} + W(x, a_j)$$

$$\psi_n^1(x, a_1) = N\left(\prod_{j=0}^{n-1} A^T(x, a_j)\right)\psi_0^1(x, a_k) \tag{19}$$

From the previous discussion, we can conclude that if we have a shape invariant superpotential $W(x, a)$ and can construct a ground function using (16), then we can solve any potential of the form:

$$V(x) = W(x, a)^2 + \frac{\hbar}{\sqrt{2m}}W'(x, a) \tag{20}$$

For instance, the harmonic oscillator can be solved by noting that the superpotential $W(x, \omega) = \frac{1}{2}\omega x$ is shape invariant:

$$W(x, \omega)^2 - \frac{\hbar}{\sqrt{2m}}W'(x, \omega)$$

[1] This restriction on $W(x)$ means that $W(x)$ is a "good" SUSY, rather than a "broken" SUSY, a distinction with important applications in theoretical physics. See for instance [9]

$$= W(x, \omega + \hbar) + \frac{\hbar}{\sqrt{2m}} W'(x, \omega + \hbar) + \hbar\omega$$

and also that:

$$V_1(x, \omega) = W(x, \omega)^2 + \frac{\hbar}{\sqrt{2m}} W'(x, \omega) = \omega x^2 - \frac{1}{2}\omega$$

The harmonic oscillator can then be solved using (18) and (19). Doing so reproduces the ladder operator method for solving the harmonic oscillator, an explanation of which can be found in [7].

It is remarkable that all well known analytically solvable potentials in nonrelativistic quantum mechanics can be solved using shape invariance. Indeed, these potentials all fall into a subset of solutions where a_i and a_{i+1} are related by a translation, $a_{i+1} = a_i + \alpha$, for some constant α. There are exactly ten types of superpotentials with this property that are also independent of $\hbar$; they are shown in Table 1. A proof that these are the only ten possible is given in [8].

Name	Superpotential
Harmonic Oscillator	Ax
Coulomb	$A - \frac{B}{x}$
3D Oscillator	$Ax - \frac{B}{x}$
Morse	$A - Be^{-x}$
Rosen-Morse I	$A\cot(x) + \frac{B}{A}$
Eckart	$A\coth(x) + \frac{B}{A}$
Rosen-Morse II	$A\tanh(x) + \frac{B}{A}$
Scarf I	$A\tan(x) - B\sec(x)$
Scarf II	$A\tanh(x) + B\text{sech}(x)$
Pöschl-Teller	$A\coth(x) - B\text{csch}(x)$

Table 1: Solvable Potentials

In Table 1, the coulomb potential can be used to solve the hydrogen atom, and the 3D oscillator can be used to solve the radial equation of the three-dimensional harmonic oscillator. The infinite square well is a special case of the Rosen-Morse I potential, and the free particle is a special case of the Rosen-Morse II potential. Using Table 1 alone, we can thus solve almost any potential found in an introductory textbook on quantum mechanics.

7 Conclusion

Supersymmetry has proved to be a useful concept in various branches of physics. In this paper, we examined the application of supersymmetry to the eigenspectra of Hamiltonians in non-relativistic quantum mechanics. This has allowed us to, for a given Hamiltonian, produce another Hamiltonian with an almost identical eigenspectra and closely related eigenvectors. For scattering states, the transmission and reflection coefficients of these Hamiltonians are also identical.

The concept of a shape invariant potential allows us to define a class of Hamiltonians which we can solve directly using supersymmetry. This concept is of particular interest as all commonly solved potentials in quantum mechanics belong in this category, suggesting that the solvability of potentials is closely related to supersymmetry. The nature of this relationship is still not fully understood, and supersymmetric quantum mechanics is still an active area of research. For instance, a general classification of all shape independent potentials has not yet been achieved.

In this paper we have only scratched the surface of possible applications of supersymmetry to quantum mechanics. Supersymmetry has been used to solve the Pauli and Dirac equations [9]. It also provides a method to produce families of isospectral Hamiltonians, which can then be applied to solving the Korteweg-de Vries equation. The WKB approximation method can be extended using supersymmetry to the SWKB method, which for many potentials produces a more accurate approximation and for the potentials in Table 10 produces exact results. Other approximation techniques have also been developed using supersymmetry, including the use of the variational principle along with supersymmetry to determine the eigenspectra of potentials. Details on the application of SUSY QM to these topics can be found in [6].

Supersymmetry has succesfully been applied to many other branches of physics other than quantum mechanics, including statistical and condensed matter physics, atomic physics, nuclear physics, particle physics, high energy physics and mathematical physics. A partial list of topics that supersymmetry has been applied to can be found in [9]. So while supersymmetry has so far failed to provide a unified theory of fundamental forces, it has proven itself to be a concept of immense utility in problems throughout physics.

References

[1] Coleman, S., and Mandula, J. (1965). All Possible Symmetries of the S Matrix. *Physics Review*, 1251-1256.

[2] Witten E. (1981). Dynamic Breaking of Supersymmetry. *Nuclear Physics B*, 513-554.

[3] Infeld, L., and Hull, T. (1951). The Factorization Method. *Reviews of Modern Physics*, 21-68.

[4] Gendenshtein, L. (1983). Derivation of exact spectra of the Schrödinger equation by means of supersymmetry. *JETP Lett.* 356-358.

[5] Nicolai, H. (1976). Supersymmetry and spin systems. *Journal of Physicals A* 1497-1506.

[6] Cooper, F. Khare, A. Sukatme, U. (2001). Supersymmetry in Quantum Mechanics.

[7] Griffiths, D. (2004). Introduction to Quantum Mechanics, 2nd Ed. 54-59

[8] Bougie, Gangopadhyaya, Mallow, and Constantin (2012). Supersymmetric Quantum Mechanics and Solvable Models. *symmetry* 452-473.

[9] Junker, G. (1995). Supersymmetric Methods in Quantum and Statistical Physics.