The WKB Method

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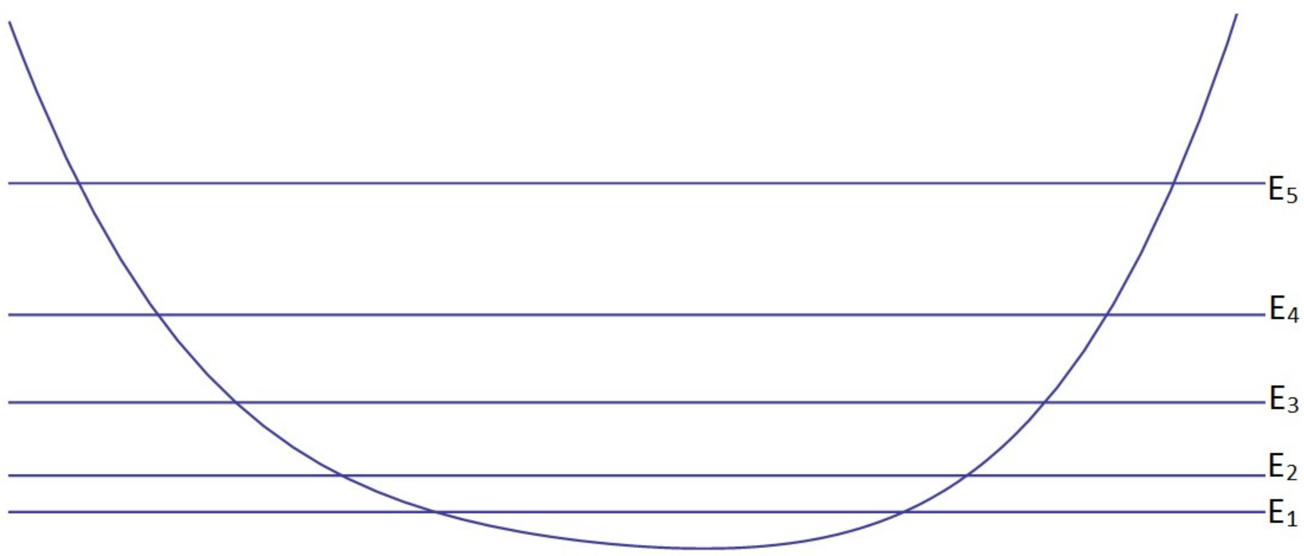
Introduction

- Time-independent Schrödinger Equation:

$$\left[-\eta^2 \frac{d^2}{dx^2} + U(x)\right]\psi(x,\eta) = E\psi(x,\eta)$$

$$\eta^2 = \frac{\hbar^2}{2m} << 1$$



Wentzel-Kramers-Brillouin (WKB) Approximation

- Semiclassical method for calculating the wavefunction
- Developed in 1926 by Wentzel, Kramers and Brillouin
- First derived by a mathematician, Jeffreys, in 1923 for general linear second order equations

WKB Approximation

- Classical Particle:

$$p(x) = \pm\sqrt{2m\left(E - U(x)\right)}$$

- For a free quantum particle

$$\psi(x) = e^{ipx/\hbar}$$

- Gives us "0th order" WKB approximation

$$\psi(x) \approx e^{i/\hbar \int p(x)dx}$$

WKB Approximation

- Classically, we want:

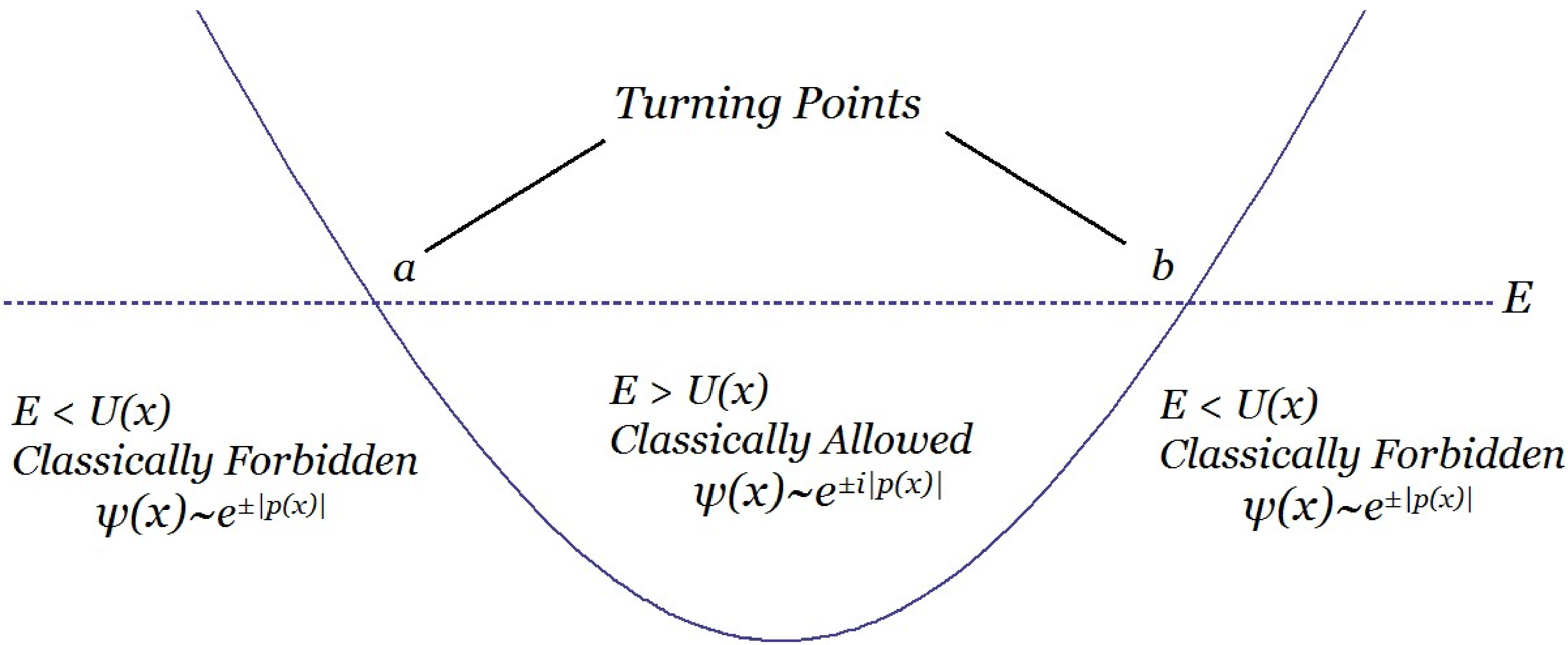
$$|\psi(x)|^2 \propto \frac{1}{|p(x)|}$$

- We get "1st order" WKB approximation

$$\psi(x) \approx \frac{1}{\sqrt{|p(x)|}} e^{i/\hbar \int p(x)\,dx}$$

Quantization Condition

- Energy Levels are the energies for which there are bounded eigenfunctions:



Quantization Condition

- Matching decaying solutions only possible if:

$$\int_a^b p(x)\,dx = \left(n - \frac{1}{2}\right)\pi\hbar$$

- Wave function has completely disappeared!
- Corresponds to Bohr-Sommerfeld quantization rule used in old quantum theory

Example: Homogenous Potential

- For a $U(x) = x^{2K}$, this can be solved to get:

$$E_n = \left(\frac{\Gamma\left(\frac{3K+1}{2K}\right)\eta\sqrt{\pi}}{\Gamma\left(\frac{2K+1}{2K}\right)} \right)^{2K/(K+1)} \left(n - \frac{1}{2}\right)^{2K/(K+1)}$$

Numerical Results

- For $U(x) = x^4$ and $\eta = 1$ we get:

$$E_n = 2.1850693 \left(n - \frac{1}{2} \right)^{4/3}$$

Eigenvalue	WBK Value	Exact Value [1]	Relative Error (%)
1	0.867145	1.060362	18
2	3.751920	3.799673	1.3
3	7.413988	7.455698	0.56
5	16.23361	16.26183	0.17
10	43.96395	43.98116	0.039
20	114.6863	114.6970	0.0093
30	199.1718	199.1799	0.0041
40	293.9418	293.949	0.0023

WKB to Higher Order

- Make substitution:

$$\psi(x,\eta) = \exp\left(\int S(x,\eta)\, dx\right)$$

- Schrödinger equation becomes:

$$S^2 + S' = \frac{U(x) - E}{\eta^2}$$

- Take power series

$$S(x,\eta) = \eta^{-1} S_{-1}(x) + S_0(x) + \eta^1 S_1(x) + \ldots$$

- First two terms give WKB approximation

WKB to Higher Order

- We get recursive relation

$$-S_{l+1} = -\frac{1}{2S_{-1}}\left(\sum_{j=0}^{l} S_j S_{l-j} + \frac{dS_l}{dx}\right)$$

- Dunham quantization condition [2]:

$$\sum_{j=0}^{\infty} \eta^{2j-1} \oint_C S_{2j-1}(z,E)\,dz = 2\pi\left(n - \frac{1}{2}\right)$$

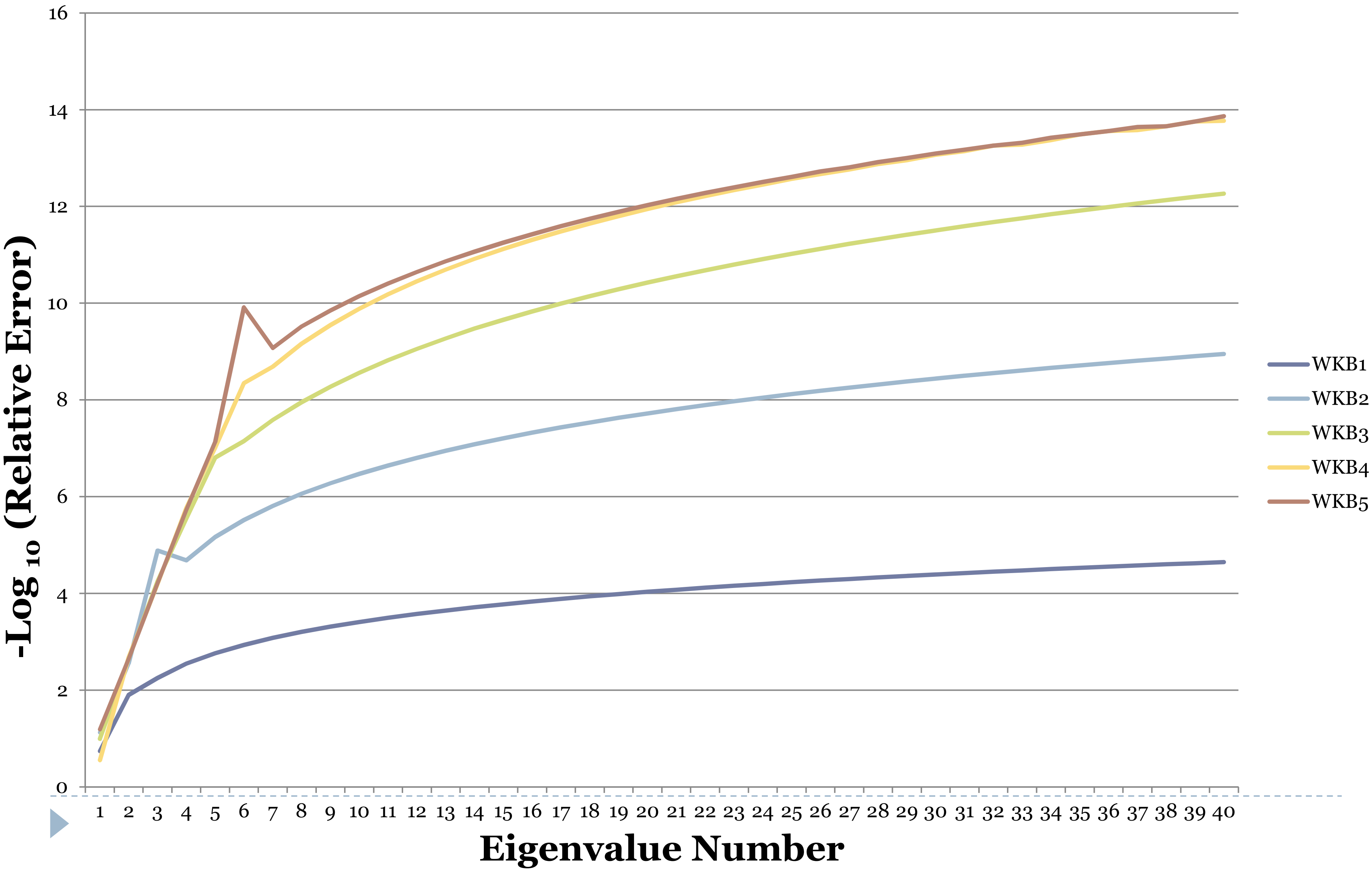
Numerical Results

- For $U(x) = x^4$ and $\eta = 1$ we get:

$$1.748\, E_n^{3/4} - 0.1498\, E_n^{-3/4} + 0.0376\, E_n^{-9/4} + 0.0939\, E_n^{-15/4} - 0.5574\, E_n^{-21/4} + \ldots = \pi\left(n - \frac{1}{2}\right)$$

Eigenvalue	WBK1	WKB3	WKB5	Exact [1]
1	0.867145	0.951643	1.128838	1.060362
Relative Error (%)	18	10	6.5	
3	7.413988	7.455282	7.455238	7.455698
Relative Error (%)	0.56	0.0056	0.0062	
10	43.96395	43.9811582184	43.981158094	43.981158097
Relative Error (%)	0.049	2.8×10^{-7}	7.2×10^{-9}	
40	293.9418	2293.9484582662	293.948458266002	293.948458266006
Relative Error (%)	0.0023	5.4×10^{-11}	1.4×10^{-12}	

Relative Error vs Eigenvalue Number

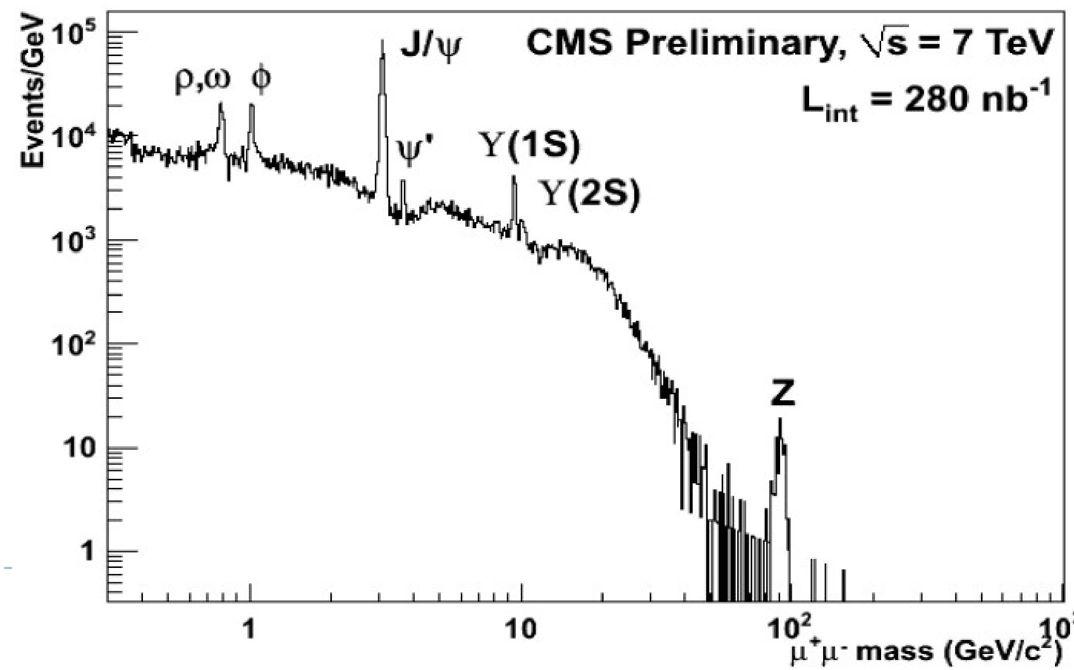


Further Applications of WKB Method

- Applicable to 3D central potentials

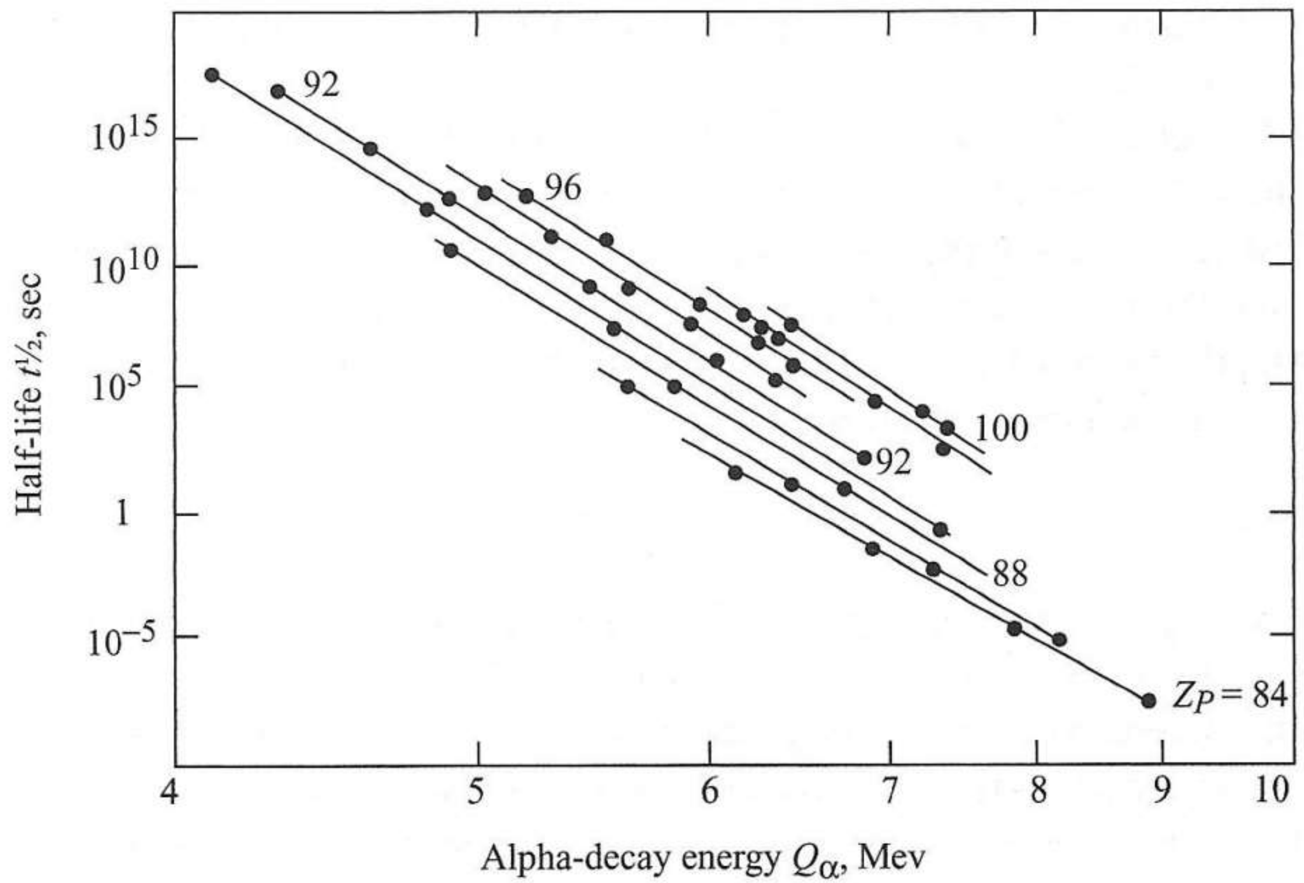
$$\left[-\eta^2 \frac{d^2}{dr^2} + U(r) + \frac{l(l+1)}{r^2}\right]\psi(r,\eta) = E\psi(r,\eta)$$

- Quarkonia Spectra [3]



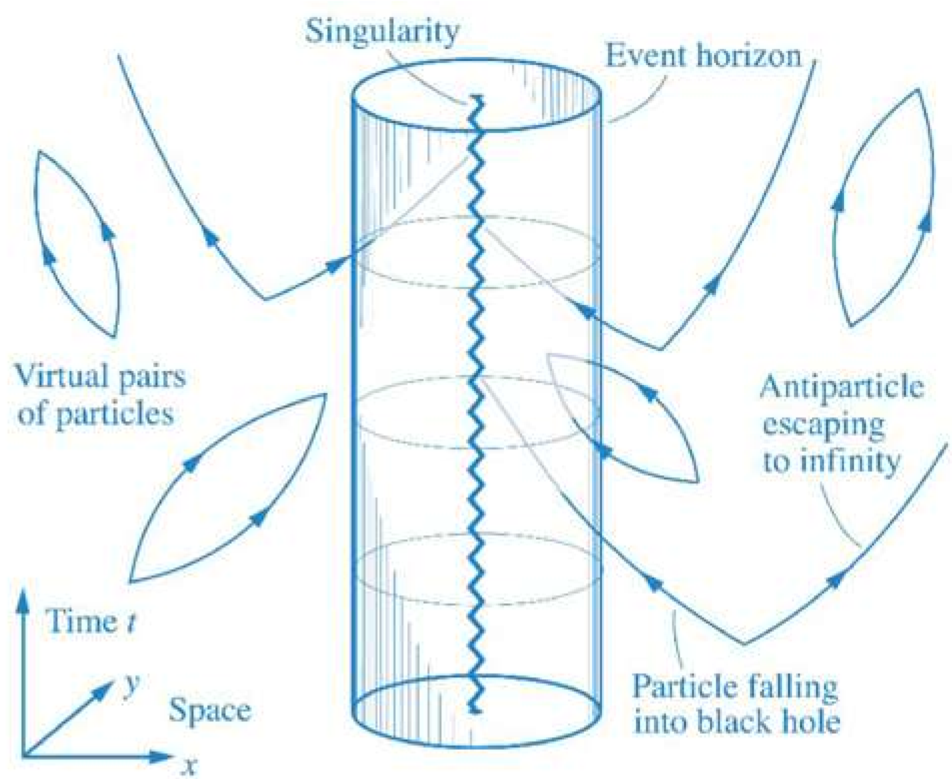
Further Applications of WKB Method

- Calculate tunnelling and reflection coefficients
- Gamow theory of alpha decay (1928) [4]



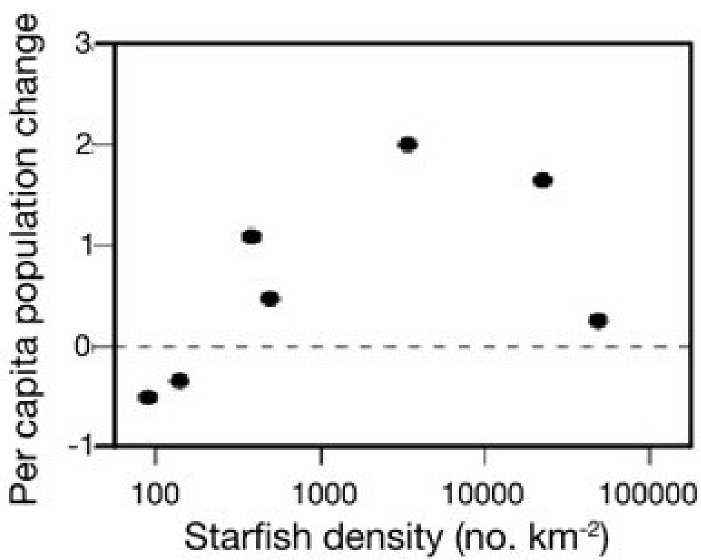
Further Applications of WKB Method

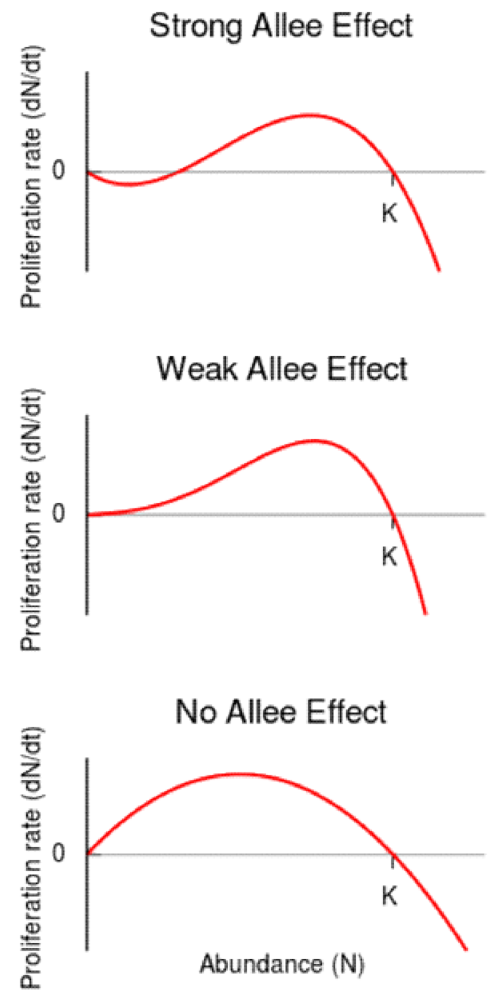
- False Vacuum Decay [5]
- Black Hole Thermodynamics [6]



Further Applications of WKB Method

- Method generalizes to other ODEs
- Inflationary Cosmology [7]
- Black Hole Dynamics [8]
- Population Dynamics [9]





Conclusion

- WKB method is a useful calculation tool
- Can be used to quickly and accurately calculate eigenvalues
- Widely applicable to many problems in physics

References

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