Higher Order Corrections to the WKB Method

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Abstract

Higher order corrections to the WKB method are explored, allowing higher order corrections to the Bohr-Sommerfeld rule to be calculated. These corrections are evaluated for potentials of the form λx^{2N}, with the first five terms explicitly found. Extensions of these results in the literature are discussed.

1 Introduction

The Wentzel-Kramers-Brillouin (WKB) method is a technique for solving the time-independent Schrödinger equation

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x)\right]\psi(x)=E\psi(x) \tag{1}$$

by taking a power series in $\hbar$. This expansion is motivated by the classical limit where $\hbar \to 0$, with the WKB method usually becoming more accurate for higher energies. When taken to first order in $\hbar$, the WKB provides a powerful method for calculating approximate eigenvalues for any potential. As $n \to \infty$, the WKB method shows that the implicit relation

$$\int_I \sqrt{E-U(x)} = \left(n-\frac{1}{2}\right)\frac{\hbar\pi}{2m} \tag{2}$$

becomes exact; the region I integrated over is the subset of $\mathbb{R}$ where $E > U(x)$. This relation is the Bohr-Sommerfeld quantization condition of old quantum theory, except with an extra $\frac{1}{2}$ on the right-hand side.

In this paper, we shall examine the higher-order corrections to (2). First we shall derive the WKB method, giving us a recursive relation to calculate corrections of arbitrary order. We shall then find the exact quantization condition, first derived by Dunham in [1]. For the case of a potential $U(x) = x^{2N}$, we derive a general expression for the n^{th} order corrections, in terms of a polynomial in N and the gamma function. With the assistance of *Mathematica*, the first few terms can be explicitly found. We finish by discussing further extensions of both the WKB method and the x^{2N} potential eigenproblem in the literature.

2 Derivation of the WKB Method

For a free particle, the Schrödinger equation is solved by the wave function $e^{\pm ipx/\hbar}$. Because of this, it seems reasonable to write a general wave function in the form

$$\psi(x,\eta) = \exp\left(\int_a^x S(x)\ dx\right), \tag{3}$$

so that if the potential varies slowly over the de Broglie wavelength of a particle, then

$$S(x) \approx \pm\frac{i}{\hbar}\sqrt{2m(E-U(x))}.$$

This is the motivation behind the WKB method, in which the above equation becomes the first term in an infinite series.

Introducing the parameter $\eta = \frac{\hbar}{\sqrt{2m}}$, we can write (1) as

$$\left[-\eta^2\frac{d^2}{dx^2} + U(x)\right]\psi(x,\eta) = E\psi(x,\eta). \tag{4}$$

By making the substitution (3), we can derive the Riccati equation

$$S^2 + S' = \frac{1}{\eta^2}(U(x) - E). \tag{5}$$

Taking a power series expansion of S in terms of η,

$$S(x,\eta) = \eta^{-1}S_{-1}(x) + S_0(x) + \eta S_1(x) + ...$$

we can then equate terms of the same power:

$$S_{-1}^2 = U(x) - E \tag{6}$$

$$2S_{-1}S_{l+1} + \sum_{j=0}^{l} S_j S_{l-j} + \frac{dS_l}{dx} = 0. \tag{7}$$

From the first equation, we deduce that to leading order,

$$S(x) = \pm\frac{i}{\hbar}\sqrt{2m(E-U(x))} + ...$$

as anticipated. Because the square root function has two branches, our choice of $S_{-1}(x)$ gives us two independent solutions to the Schrödinger equation, as can be expected for a second-order equation. So after choosing the branch of $S_{-1} = \pm\sqrt{U(x)-E}$, we can determine each S_{l+1} recursively by the formula

$$S_{l+1} = -\frac{1}{2S_{-1}}\left(\sum_{j=0}^{l} S_j S_{l-j} + \frac{dS_l}{dx}\right). \tag{8}$$

To examine the relationship between the two possible solutions, we can use superscript $\pm$ to distinguish the two, so that we find S_k^+ for the case of $S_{-1}^+ = +\sqrt{U(x)-E}$. Then since

(7) holds for S_k^+, we note that these equations also hold for $(-1)^k S_k^+$. The first term of this latter series of functions is $(-1)^{-1} S_{-1}^+ = S_{-1}^-$, and so we can deduce that

$$S_k^- = (-1)^k S_k^+.$$

Therefore changing the sign of S_{-1} changes only the sign of the odd terms. We will from now on drop the superscripts, setting $S_k = S_k^+$.

Let S_{even} and S_{odd} denote the sum of even and odd powers of η respectively. Using (7) we can write

$$-\frac{dS_{2l+1}}{dx} = 2S_{-1}S_{2l+2} + \sum_{j=0}^{2l+1} S_j S_{l-j} = 2l+1\text{-term of } 2S_{\text{odd}}S_{\text{even}}$$

and so

$$-\frac{dS_{\text{odd}}}{dx} = 2S_{\text{odd}}S_{\text{even}}.$$

This allows us to write S_{even} as the total derivative

$$S_{\text{even}} = -\frac{1}{2}\frac{d}{dx}\log S_{\text{odd}} \tag{9}$$

giving us the wave function

$$\psi = \frac{C}{\sqrt{|S_{\text{odd}}|}} \exp\left(\int_a^x S_{\text{odd}}\ dx\right).$$

As a result, we can write two independent solutions of the equation:

$$\psi_\pm = \frac{C_\pm}{\sqrt{|S_{\text{odd}}|}} \exp\left(\pm\int_a^x S_{\text{odd}}\ dx\right) \tag{10}$$

since the overall normalization factor of these solutions is arbitrary.

Examining (8), we find that each S_k in the series has an additional factor of S_{-1}^{-1}, so that when $|S_{-1}|$ is large the higher order terms will be small. Since

$$S_{-1} = \sqrt{U(x) - E}$$

this means that correction will be small both when $E >> U(x)$ and when $U(x) >> E$. On the other hand, if $U(a) = E$ at some point a, then our expression for S_k will become singular. Taking the classical limit, a corresponds to a turning point, since it marks the boundary between the region where $U(x) < E$, which is the classically allowed region of phase space, and $U(x) > E$ which classically is forbidden. For the classical region,

$$S_{-1} = \pm i\sqrt{E - U(x)}$$

is purely imaginary, as indeed S_{odd} is, so that the wave function is oscillatory in this region. In the forbidden region however,

$$S_{-1} = \pm\sqrt{U(x) - E}$$

is real, so that S_{odd} is also real. This gives us a exponentially growing and decaying solution to the Schrödinger equation. To find the eigenvalues of a potential, we will need to connect the wave function in these two different regions, a problem we shall solve in the next section.

Before doing so however, it is important to ask in what sense our series for $S(x)$ converges. In fact, it does not converge for any finite value of x. Instead, our expression for $S(x)$ is asymptotically convergent both for small η and for large $|x|$. This means that as we take η to zero, the error from truncating at the n^{th} term is of order η^n. So our series can be used to give good approximations to $S(x)$ by truncating at a well chosen n, in spite of the divergence of our series as $n \to \infty$. Further discussion of this behavior can be found in many textbooks, for instance, [2].

3 The Energy Level Condition

Our derivation of the energy level condition follows [1]. We shall consider the simplest possible case of a concave and analytic potential $U(x)$, going to infinity as $x \to \pm\infty$. Then for every energy E there will be two turning points, a and b. For E to be an eigenstate, we require that there is a solution $\psi(x)$ of (1) for this energy which bounded along the real axis. As $U(x)$ is analytic, so is $\psi(x)$, allowing us to extend the wave function onto the complex plane. We can then demand that ψ be single-valued. Finally, any solution of (1) can be taken to be real along the real axis. The Schwarz reflection principle then dictates that $\psi(z^*) = \psi(z)^*$.

We shall divide the real axis into three regions; region I lies to the left of a, region II in between a and b, and region III lies to the right of b. We shall also take the branch cut of $S_{-1}(x) = \sqrt{U(x) - E}$ to be along the real axis between a and b. This allows us to circumvent the singularities at a and b by taking contours around these points, though we cannot pass through the real axis between a and b because of the branch cut.

If our wave function is to be bounded on the real axis, it must decay when $x < a$. We therefore demand that in region I,

$$\psi_I(x) = C_1 \exp\left(-\int_x^{x_1} S(x)\ dx\right)$$

where x_1 is a point to the left of a. Our choice of x_1 is arbitrary, since any change can be absorbed into the normalization constant C.

In region II we require the wave function to be real, so we must use both branches of $S_{-1}(x)$. The most general form of ψ_{II} is then

$$\psi_{II}(x) = C_2 \exp\left(\int_x^{x_2} S(y)\ dy\right) + C_2^* \exp\left(-\int_x^{x_2} S(y)\ dy\right)$$

where x_2 is between a and b. Passing from x_1 to x_2 in the upper-half of the complex plane, we find that

$$C_1 \exp\left(-\int_x^{x_1} S(x)\ dx\right) \to C_1 \exp\left(-\int_{x_2}^{x_1} S(x)\ dx\right) \exp\left(-\int_x^{x_2} S(x)\ dx\right).$$

If instead we follow a contour passing underneath a, then we will get the conjugate term, and hence

$$\psi_{II}(x) = C_1 \left(e^{\int_{x_1}^{x_2} S(x)\ dx} e^{-\int_x^{x_2} S(x)\ dx} + e^{\int_{x_1}^{x_2} S^*(x)\ dx} e^{\int_x^{x_2} S(x)\ dx}\right)$$

where the integration takes a contour on the upper-half of the complex plane. The transformation of the coefficients in our expression is known as Stokes phenomenon, and is a result of the asymptotic nature of the WKB expansion.

If our solution is to be bounded along the real axis, in region III the wave function must have form

$$\psi_{III}(x) = C_3 \exp\left(-\int_{x_3}^{x} S(x)\ dx\right)$$

for real C_3 and $x_3 > b$. We can extend this leftward to region II in the same method as before, so that

$$\psi_{II}(x) = C_3\left(e^{\int_{x_2}^{x_3} S(x)\ dx} e^{\int_{x}^{x_2} S(x)\ dx} + e^{\int_{x_2}^{x_3} S^*(x)\ dx} e^{-\int_{x}^{x_2} S(x)\ dx}\right).$$

Matching this with our other expression for $\psi_I I(x)$ requires that

$$C_3 e^{\int_{x_2}^{x_3} S(x)\ dx} = C_1 e^{\int_{x_1}^{x_2} S^*(x)\ dx}.$$

Since C_1 and C_3 are real, this requires that

$$\operatorname{Im}\left(\int_{x_2}^{x_3} S(x)\ dx - \int_{x_1}^{x_2} S^*(x)\ dx\right) = n\pi$$

for some $n \in \mathbb{Z}$. The left-hand side can be rewritten as

$$\operatorname{Im}\left(\int_{x_2}^{x_3} S(x)\ dx - \int_{x_1}^{x_2} S^*(x)\ dx\right) = \operatorname{Im}\left(\int_{x_2}^{x_3} S(x)\ dx + \int_{x_1}^{x_2} S(x)\ dx\right)$$

$$= \frac{1}{2}\left(\int_{x_1}^{x_3} S(x)\ dx + \left(\int_{x_1}^{x_3} S(x)\ dx\right)^*\right) = \frac{1}{2}\oint S(z)\ dz.$$

The contour in the last equality enclosed the two turning points a and b. We finally have the condition

$$\oint S(z)\ dz = 2\pi n.$$

To simplify further, we can apply (9) to write

$$\oint S(z)\ dz = \oint S_{\text{odd}}(z) - \frac{1}{2}\frac{d}{dz}\log S_{\text{odd}}(z)\ dz.$$

Since the sign of $S_{\text{odd}}(z)$ change from region I to region III, the second integral is nonzero:

$$\oint \frac{1}{2}\frac{d}{dz}\log S_{\text{odd}}(z)\ dz = \frac{1}{2}\left(\int_{x_1}^{x_3} \frac{d}{dx}\log S_{\text{odd}}(x)\ dx + \left(\int_{x_1}^{x_3} \frac{d}{dx}\log S_{\text{odd}}(x)\ dx\right)^*\right)$$

$$= \operatorname{Im}\left(\log(S(x_3)) - \log(S(x_1))\right) = \pi.$$

So we find that for E to be an eigenvalue of our problem, the equation

$$\oint S_{\text{odd}}(z)\ dz = 2\pi\left(n + \frac{1}{2}\right) \tag{11}$$

must be satisfied. Because of the asymptotic nature of the WKB expansion, our eigenvalue condition will in general give us an asymptotic expansion.

4 Eigenvalues of the Potential λx^{2N}

We shall now evaluate the terms of (11) for potentials of the form $U(x) = \lambda x^{2N}$, where N is a positive integer. The simplicity of evaluating the terms in (11) for these potentials is a result of their scaling behavior. We can use this to eliminate λ by making a substitution $y = \lambda^{1/2N} x$ in (4). This gives us

$$\left[-(\eta\lambda^{1/N})^2 \frac{d^2}{dy^2} + y^{2N}\right] \psi = E\psi.$$

Therefore solving the eigenvalues of the potential λx^{2N} with a given η is equivalent to solving the potential x^{2N} with $\eta\lambda^{1/2N}$. So for the moment we shall set $\lambda = 1$.

Let us denote $S_k(x, E)$ to be the function $S_k(x)$ at a particular energy E and with $\lambda = 1$. Then we find that S_k scales with energy according to the relationship

$$S_k(x, E) = E^{-1/2N - k(1/2+1/2N)} S_k(xE^{-1/2N}, 1). \tag{12}$$

This can be proved by first confirming it in the case of $k = -1$

$$S_{-1}(x, E) = \sqrt{x^{2N} - E} = E^{1/2}\sqrt{\frac{x^{2N}}{E} - 1} = E^{1/2} S_{-1}(xE^{-1/2N}, 1).$$

By applying (8), we can then prove that if (12) holds for all $k \le l$, then it also holds for $l+1$:

$$S_{l+1}(x, E) = -\frac{1}{2S_{-1}(x, E)} \left(\sum_{j=0}^{l} S_j(x, E) S_{l-j}(x, E) + \frac{dS_l(x, E)}{dx} \right).$$

$$= -\frac{E^{-1/2N - l(1/2+1/2N)}}{2E^{1/2} S_{-1}(xE^{-1/2N}, 1)} \left(E^{-1/2N} \sum_{j=0}^{l} S_j(xE^{-1/2N}, 1) S_{l-j}(xE^{-1/2N}, 1) + \frac{dS_l(xE^{-1/2N}, 1)}{dx} \right)$$

$$= -\frac{E^{-1/2 - 1/N - l(1/2+1/2N)}}{2S_{-1}(xE^{-1/2N}, 1)} \left(\sum_{j=0}^{l} S_j(xE^{-1/2N}, 1) S_{l-j}(xE^{-1/2N}, 1) + \left.\frac{dS_l(y, 1)}{dy}\right|_{y = E^{-1/2N}x} \right)$$

$$= E^{-1/2N - (l+1)(1/2+1/2N)} S_{l+1}(xE^{-1/2N}, 1).$$

So by induction, (12) holds for every k.

Substituting (12) into (11), we find that

$$\oint S_{\text{odd}}(z, E)\, dz = \oint \sum_{k=-1}^{\infty} E^{-1/2N - (2k+1)(1/2+1/2N)} \eta^{2k+1} S_{2k+1}(zE^{-1/2N}, 1)\, dz$$

$$= \sum_{k=-1}^{\infty} E^{-\frac{(k+1)}{N} - k - \frac{1}{2} + \frac{1}{2N}} \eta^{2k+1} \oint S_{2k+1}(z, 1)\, dz = E^{1/2N + 1/2} \sum_{k=0}^{\infty} \frac{\eta^{2k-1}}{E^{k(1+1/N)}} \oint S_{2k-1}(z, 1).$$

If we label

$$A_k(N) = \int S_{2k-1}(z, 1)\, dz,$$

we can then write the quantization condition for a general η and λ as

$$\frac{E^{1/2N+1/2}}{\lambda^{1/2N}\eta}\sum_{k=0}^{\infty}\left(\frac{\eta^2\lambda^{1/N}}{E^{1+1/N}}\right)^k A_k(N) = 2\pi\eta\left(n+\frac{1}{2}\right). \tag{13}$$

All that is left is to calculate $A_k(N)$. To do so we can recursively evaluate (8) to find $S_{2k-1}(z)$. In general, we find that

$$S_k(z) = \frac{p_k(z^{2N})}{2^{k+1}z^{k+1}(1-z^{2N})^{3k/2+1}} \tag{14}$$

where $p_k(z)$ is a polynomial of order at most $k+1$, and furthermore where $p_k(z^{2N})/z^{k+1}$ is a polynomial.

To prove this, first note that it is true for $k=-1$, since

$$S_{-1}(z) = \sqrt{1-z^{2N}}.$$

Now if we assume that our formula holds for all $k<l$, then applying (8),

$$\begin{aligned}
S_{l+1}(z) = -\frac{1}{2\sqrt{1-z^{2N}}}\Bigg(&\frac{1}{2^{l+2}z^{l+2}(1-z^{2N})^{3l/2+2}}\sum_{j=0}^{l}p_j(z^{2N})p_{l-j}(z^{2N}) \\
&+\frac{((-l-1)p_l(z^{2N})+2Nz^{2N}p_l'(z^{2N}))(1-z^{2N})+2N(3l/2+1)z^{2N}p_l(z^{2N})}{2^{l+1}z^{l+2}(1-z^{2N})^{3l/2+2}}\Bigg) \\
= -\frac{\frac{1}{2}\sum_{j=0}^{l}p_j(z^{2N})p_{l-j}(z^{2N})+(2Nz^{2N}p_l'(z^{2N})-(l+1)p_l(z^{2N}))(1-z^{2N})+N(3l+2)z^{2N}p_l(z^{2N})}{2^{l+2}z^{l+2}(1-z^{2N})^{3(l+1)/2+1}}&.
\end{aligned}$$

So if we define

$$p_{l+1}(z) = \frac{1}{2}\sum_{j=0}^{l}p_j(z)p_{l-j}(z)+(2Nzp_l'(z)-(l+1)p_l(z))(1-z)+N(3l+2)zp_l(z)$$

then by induction, (14) always holds. Furthermore the degree of $p_{l+1}(z)$ will at most be $l+1$, and $p_{l+1}(z^{2N})/z^{l+2}$ will also be a polynomial.

We have now simplified the task of calculating A_k to that of calculating integrals of the form

$$I(N,m,n) = \oint \frac{z^m}{\left(\sqrt{1-z^{2N}}\right)^n}\,dz$$

around a contour enclosing the real axis between -1 and 1. To evaluate this integral, we can make the substitution $w=z^{2N}$, so that

$$I(N,m,n) = \oint \frac{w^{m/2N+1/2N-1}}{2N\left(\sqrt{1-w}\right)^n}\,dw.$$

If $n<2$, then by taking the contour to be arbitrarily close to the line $[0,1]$, we find that

$$I(N,m,n) = 2\int_0^1 \frac{w^{m/2N+1/2N-1}}{2N\left(\sqrt{1-w}\right)^n}\,dw, \quad n<2$$

since the contours around the points 0 and 1 will disappear, and the branch cut of $\sqrt{1-w}$ means that the two integrals along the real axis add together. This type of integral is known as an Eulerian Integral of the First Kind, and can be expressed in terms of the gamma function

$$I(N,m,n) = 2\int_0^1 \frac{w^{m/2N+1/2N-1}}{2N\left(\sqrt{1-w}\right)^n}\,dw = \frac{\Gamma(\frac{m+1}{2N})\Gamma(1-\frac{n}{2})}{N\Gamma(\frac{m+1}{2N}+1-\frac{n}{2})}, \quad n<2.$$

But from our definition of $I(N,m,n)$, we know it is analytic in n, so that

$$I(N,m,n) = \frac{\Gamma(\frac{m+1}{2N})\Gamma(1-\frac{n}{2})}{N\Gamma(\frac{m+1}{2N}+1-\frac{n}{2})}$$

should hold for every value of n.

So in principle we can now evaluate each coefficient A_k in terms of the gamma function. Using (14), we find that

$$A_k(N) = \oint \frac{p_{2k-1}(z^{2N})}{2^{2k}z^{2k}(1-z^{2N})^{3(2k-1)/2+1}}\,dz = \sum_{j=1}^{2k}\oint \frac{a_j(N)z^{2(jN-k)}}{2^{2k}(1-z^{2N})^{3(2k-1)/2+1}}\,dz$$

$$= \sum_{j=1}^{2k} \frac{a_j(N)\Gamma(j+\frac{1-2k}{2N})\Gamma(\frac{-3(2k-1)}{2})}{2^{2k}N\Gamma\left(j+(1-2k)\left(\frac{1}{2N}+\frac{3}{2}\right)\right)}$$

where a_j is the j-th coefficients of $p_{2k-1}(z)$. This expression can be furthered simplified to

$$A_k(N) = \frac{2\sqrt{\pi}\Gamma(1+\frac{1-2k}{2N})P_k(N)(-1)^k}{\Gamma(\frac{3-2k}{2}+\frac{1-2k}{2N})(2k+2)!2^n} \tag{15}$$

where $P_k(N)$ is a polynomial in N [3].

The first few polynomials are given below

$$P_0(N) = 1$$

$$P_1(N) = 2(2N-1)$$

$$P_2(N) = (2N-3)(2N-1)(4N+3)$$

$$P_3(N) = \frac{4}{9}(2N-5)(2N-1)(196N^3+88N^2-234N-139)$$

$$P_4(N) = \frac{1}{3}(2N-7)(2N-1)(13824N^5-4800N^4-41504N^3-6484N^2+29432N+12961)$$

They were calculated using *Mathematica*. We can see that the higher order terms become increasingly complicated. There is no known simple formula for the polynomials, although a number of special values of the polynomials have been found. A list can be found in [3].

The gamma function has a singularity at each of the non-positive integers. Because of this, in (15), if

$$R = \frac{1-2k}{2N} + \frac{3-2k}{2}$$

is a non-positive integer, then $A_k(N)$ will be zero. First note that R must be negative if $k > 1$ or if $k = N = 1$. Since N and k are integers, we find that R will be an integer iff

$$\frac{1-2k}{2N} \in \mathbb{Z} + \frac{1}{2}$$

which can be rewritten as

$$\frac{1-2k}{N} \equiv 1 \mod 2. \tag{16}$$

If this relationship is satisfied, then $A_k(N) = 0$.

If $N = 1$, then this (16) is satisfied for every k, and so only A_1 is nonzero. The exact energy levels therefore satisfy the relation

$$E_n \frac{2\sqrt{\pi}\ \Gamma(1+\frac{1}{2})}{\eta\sqrt{\lambda}\Gamma(2)} = 2\pi\left(n+\frac{1}{2}\right),$$

so that

$$E_n = \frac{\hbar\sqrt{2\lambda}}{\sqrt{m}}\left(n+\frac{1}{2}\right).$$

This gives us the correct energy levels for the harmonic oscillator.

If N is odd, then it is equal to $2r-1$ for some integer r. We then find that $A_{r+aN}(N)$ will be zero, where $a \in \mathbb{N}$, giving us an infinite number of zero terms. On the other hand, if N is even (16) cannot be satisfied, since the numerator is odd but the denominator is even.

Since the coefficients $A_k(N)$ are easy to calculate, (13) provides an efficient way to numerically calculate the eigenvalues of an x^{2N} potential, since equation solving can be quickly achieved numerically. In the three tables below, the first ten eigenvalues are calculated for the potentials $N = 2, 5$, and 10, taking the 1^{st} through to 5^{th} order terms. These can be compared to the eigenvalues calculated through a Numerov algorithm, which are exact to the level of precision given.

In the tables we can see that the accuracy of the WKB method increases with higher eigenvalues. It generally increases with the order of the method as well, though since the WKB series is asymptotic, the accuracy only increases up to a point. Indeed, for the x^{10} potential the first eigenvalue cannot be calculated to 5^{th} order because the WKB series is always larger than $\frac{\pi}{2}$ in this case. We also find that the method is most accurate for $N = 2$, and gets worse as N increases. This is unsurprising, considering that the size of each term in the WKB series increases with N.

1^{st} order	2^{nd} order	3^{th} order	4^{th} order	5^{th} order	Exact	Relative Error
0.867	0.980766	0.95164303	0.765104328	1.128836370	1.060362090	0.065 (5)
3.752	3.810330	3.80837726	3.807719246	3.808244858	3.799673029	0.0021 (4)
7.414	7.455796	7.45528245	7.455219387	7.455236578	7.455697938	1.3×10^{-5} (4)
11.61	11.64499	11.6447787	11.64476180	11.64476241	11.64474551	1.4×10^{-6} (4)
16.23	16.26194	16.2618286	16.26182443	16.26181215	16.26182601	9.7×10^{-8} (4)
21.21	21.23844	21.2383744	21.23837281	21.23837291	21.23837291	1.5×10^{-10} (5)
26.51	26.52851	26.5284719	26.52847112	26.52847115	26.52847118	1.1×10^{-9} (5)
32.08	32.09863	32.0985981	32.09859768	32.09859769	32.09859771	6.4×10^{-10} (5)
37.90	37.92302	37.9230012	37.92300100	37.92300101	37.92300102	3.2×10^{-10} (5)
43.96	43.98117	43.9811582	43.98115807	43.98115808	43.98115809	2.7×10^{-10} (5)

Table 1: First ten eigenvalues of x^4 potential calculated with 1-5^{th} order WKB method, compared to the exact eigenvalue. The relative error is the error between the exact value and the closest value calculated through the WKB order; the quantity in brackets is the order of the WKB method used.

1^{st} order	2^{nd} order	3^{th} order	4^{th} order	5^{th} order	Exact	Relative Error
0.737	1.22974	1.403314238	1.4033142	N/A	1.298843696	0.0532 (2)
4.596	4.99918	5.034469044	5.0344690	4.8631622	5.097876534	0.0124 (3)
10.77	11.1150	11.12698931	11.126989	11.119336	11.15431824	0.0024 (3)
18.86	19.1780	19.18364737	19.183647	19.182674	19.18880960	2.7×10^{-4} (3)
28.68	28.9681	28.97123279	28.971233	28.971029	28.97146725	8.1×10^{-6} (3)
40.07	40.3408	40.34277076	40.342771	40.342713	40.34261578	2.4×10^{-6} (5)
52.94	53.1910	53.19237527	53.192375	53.192355	53.19230657	9.1×10^{-7} (5)
67.19	67.4373	67.43823466	67.438235	67.438226	67.43821500	1.7×10^{-7} (5)
82.78	83.0134	83.01429386	83.014294	83.014290	83.01429156	1.7×10^{-8} (5)
99.64	99.8650	99.86555915	99.865559	99.865557	99.86556674	7.6×10^{-8} (3)

Table 2: First ten eigenvalues of x^{10} potential calculated with 1-5^{th} order WKB method, compared to the exact eigenvalue.

1^{st} order	2^{nd} order	3^{th} order	4^{th} order	5^{th} order	Exact	Relative Error
0.680	1.7595	2.70925	4.59141	7.32461	1.56050853	0.13 (2)
5.015	6.1158	6.54339	7.47529	9.43027	6.219361109	0.017 (2)
12.70	13.737	13.9220	14.1684	14.7453	13.91315974	6.3×10^{-4} (3)
23.41	24.399	24.4945	24.5666	24.6746	24.54991597	6.8×10^{-4} (4)
36.97	37.918	37.9746	38.0014	38.0267	38.02437186	6.1×10^{-5} (5)
53.24	54.162	54.1994	54.2113	54.2189	54.23167781	2.4×10^{-4} (5)
72.14	73.032	73.0579	73.0639	73.0667	73.07628855	1.3×10^{-4} (5)
93.57	94.446	94.4658	94.4691	94.4703	94.47557816	5.6×10^{-5} (5)
117.49	118.340	118.355	118.357	118.357	118.3596604	2.2×10^{-5} (5)
143.82	144.655	144.667	144.668	144.668	144.6694223	8.5×10^{-6} (5)

Table 3: First ten eigenvalues of x^{20} potential calculated with 1-5^{th} order WKB method, compared to the exact eigenvalue.

5 Generalizations and Conclusion

We have discussed the WKB method to all orders, and derived the first few higher-order corrections to the quantization rule for the case of a potential x^{2N}. There are a number of extensions to these results in the literature.

For certain special potentials, the exact quantization condition can be used to give a convergent series which can be used to solve the eigenvalues of the potential. We have seen that this is the case for the harmonic oscillator, since all but the lowest order term are zero. This is also the case for the infinite square well, Coulomb potential, and Morse potential [4], though a modification to the right-hand side of (2) are necessary for the first two cases to take into account the singular nature of these potentials. For the Coulomb potentials these are known as Langer corrections [5]. For the $-V_0(\cosh x)^{-2}$ potential [3], the $\tan^2(x)$ potential, and the Pöschl-Teller potential [6], the series is convergent and can be summed to calculate the eigenvalues. In general, the conditions required for the series to converge are quite restrictive, and are given in [7]. The WKB method has been extended to supersymmetric quantum mechanics, leading to a method, the supersymmetric WKB (SWKB), which gives the exact quantization to lowest order for all simple shape-invariant potentials [8]. This class includes all of the aforementioned potentials.

Given our asymptotic sequence for the energy eigenvalues, it is possible to re-sum the series in order to get a convergent series through the use of spectral determinants and zeta functions [9][10]. This analysis can also be extended to the case of potentials of odd degree $|x|^{2n+1}$ [11]. Remarkably, spectral problems for x^{2N} potentials have been associated with the ground state eigenvalues of integrable models in $1+1$ dimensional quantum fields theories [12][13]. This relationship is part of a more general correspondence known as the ordinary differential equation/integral model (ODE/IM) connection, connecting integrable models to certain differential equations studied in the complex plane [14].

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