Wigner Functions

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Introduction

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Can't we just use a probability distribution $P(x,p)$ ***like we did in classical mechanics?***

Classical Ensembles

Classically, *P(x,p)* can represent a classical ensemble.

We can marginalise:

$$P(x) = \int_{-\infty}^{\infty} P(x, p)\,dp$$

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$$\langle A(x,\,p)\rangle = \int_{-\infty}^{\infty} A(x,\,p)\,P(x,\,p)\,dp\,dx$$

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What do we get if we formulate Quantum Mechanics in the same way?

Introducing the Wigner Function

The function we are looking for is the Wigner function:

$$W(x,p)=\frac{1}{\pi}\int\langle x+y|\psi\rangle\langle\psi|x-y\rangle e^{2ipy}dy$$

We set $\hbar=1$ in the equation.

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We set $\hbar=1$ in the equation.

For a mixed state:

$$W(x,p)=\frac{1}{\pi}\int\langle x+y|\hat{\rho}|x-y\rangle e^{-2ipy}dy$$

Properties of the Wigner Function

1. Symmetric in *x* and *p*:

$$W(x,p)=\frac{1}{\pi}\int\left\langle p+q\left|\hat{\rho}\right|p-q\right\rangle e^{2ixq}dq$$

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Properties of the Wigner Function

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4. Some expectation values:

$$\int f(x)W(x,p)dp = f(x)\langle x|\hat{\rho}|x\rangle = \langle f(\hat{x})\rangle$$

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Can we do this in general?

Operator Representations

We calculate

$$\frac{1}{\pi}\int \left\langle x+y\left|\hat{x}\hat{\rho}\right|x-y\right\rangle e^{-2ipy}\,dy$$

and find:

$$\hat{x}\hat{\rho}\longrightarrow\left(x+\frac{i}{2}\frac{\partial}{\partial p}\right)W$$

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and find:

$$\hat{x}\hat{\rho} \longrightarrow \left(x+\frac{i}{2}\frac{\partial}{\partial p}\right)W$$

By symmetry:

$$\hat{p}\hat{\rho} \longrightarrow \left(p-\frac{i}{2}\frac{\partial}{\partial x}\right)W$$

Operator Representations

Conjugation give us the other two relations:

$$\hat{x}\hat{\rho} \longrightarrow \left(x + \frac{i}{2}\frac{\partial}{\partial p} \right) W \qquad \hat{p}\hat{\rho} \longrightarrow \left(p - \frac{i}{2}\frac{\partial}{\partial x} \right) W$$

$$\hat{\rho}\hat{x} \longrightarrow \left(x - \frac{i}{2}\frac{\partial}{\partial p} \right) W \qquad \hat{\rho}\hat{p} \longrightarrow \left(p + \frac{i}{2}\frac{\partial}{\partial x} \right) W$$

These are the operator correspondences.

We can now represent $\hat{g}(\hat{x}, \hat{p})$ for any $\hat{g}$.

Calculating Expectation Values

We can calculate

$$\left\langle (\hat{x}\hat{p} + \hat{p}\hat{x})\hat{\rho} \right\rangle = \int xpW(x, p)dxdp$$

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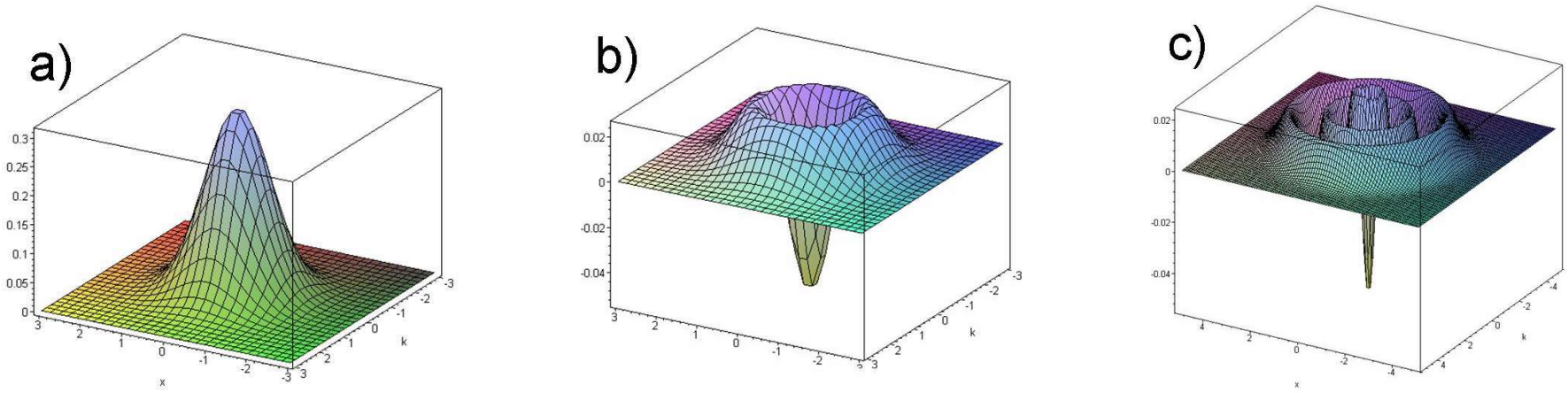
In general:

$$\left\langle :\hat{x}^m \hat{p}^n :_{sym} \hat{\rho} \right\rangle = \int x^m p^n W(x,p)dxdp$$

What is the Wigner function?

The Wigner function can be negative! It not a probability distribution.

Below is the ground state (left), first excited state (centre), and fifth excited state (right) of the Harmonic Oscillator.



What is the Wigner function?

The Wigner function can be negative! It not a probability distribution.

It is a quasi-probability distribution.

Negative areas have to be small. Wigner function still must satisfy uncertainty relationship.

Evolving the Wigner Function

The quantum Liouville equation is

$$\frac{d\hat{\rho}}{dt} = i[\hat{\rho}, \hat{H}]$$

The operator correspondences mean we can convert this to a PDE.

This also works for open systems:

$$\frac{d\hat{\rho}}{dt} = i[\hat{\rho}, \hat{H}] + \kappa \mathcal{D}[\hat{c}]$$

The Correspondence Principle

Wigner function allows formulation of QM that is almost identical to CM.

It is ideal for studying the Classical-Quantum Limit.

We will find a novel approximation method for simulating Wigner functions.

The Correspondence Principle

Let's restore the Planck's constants in our equations. The operator correspondences become:

$$\hat{x}\hat{\rho} \longrightarrow \left(x + \frac{i\hbar}{2}\frac{\partial}{\partial p} \right) W \qquad \hat{p}\hat{\rho} \longrightarrow \left(p - \frac{i\hbar}{2}\frac{\partial}{\partial x} \right) W$$

In the classical limit, these operators become *x* and *p*.

The Correspondence Principle

Calculate the commutator:

$$[\hat{x}^m \hat{p}^n, \hat{x}^a \hat{p}^b] \longrightarrow \left[\left(x + \frac{i\hbar}{2} \frac{\partial}{\partial p} \right)^m \left(p - \frac{i\hbar}{2} \frac{\partial}{\partial x} \right)^n, \left(x + \frac{i\hbar}{2} \frac{\partial}{\partial p} \right)^a \left(p - \frac{i\hbar}{2} \frac{\partial}{\partial x} \right)^b \right]$$

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Expand as a power series in ħ.

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Expand as a power series in ħ. Zeroth order term vanishes.

First order term is:

$$i\hbar(mb - na) x^{m+a-1} p^{n+b-1}$$

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First order term is:

$$i\hbar(mb - na)x^{m+a-1} p^{n+b-1} = i\hbar\{x^m p^n, x^a p^b\}$$

which means that:

$$[\hat{A}(x,p), \hat{B}(x,p)] = i\hbar\{A(x,p), B(x,p)\} + O(\hbar^2)$$

Liouville's Equation

Liouville's equation:

is to first order: $$\frac{d\hat{\rho}}{dt} = \frac{i}{\hbar}[\hat{\rho}, \hat{H}]$$

$$\frac{dW}{dt} = \frac{i}{\hbar}\left(i\hbar\{W, H\} + O(\hbar^2)\right)$$

$$= -\{W, H\} + O(\hbar)$$

To first order, the Wigner function evolves like a classical probability distribution.

Simulating the Wigner Equation

Since Wigner function evolves classically, we can use simulate it classically:

$$\dot{q} = \frac{\partial H}{\partial p} \quad \dot{p} = -\frac{\partial H}{\partial q}$$

We sample points, simulate them, and then calculate expectation values.

Good for large systems.

Fokker-Planck Equation

To second order:

$$\frac{dW}{dt} = \left(\frac{\partial}{\partial x_i} A_i + \frac{\partial}{\partial x_i \partial x_j} B_{ij} \right) W$$

where x_1 is the position, x_2 is the momentum and we sum over the indices.

Fokker-Planck Equation

We want to solve Fokker-Planck equation like before. Can we write:

$$dx_i = A_i dt + ?$$

and then average over the paths?

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and then average over the paths?

Can't do this with an ODE.

Instead we use non-deterministic paths.

Diffusion

The Diffusion Equation is a special case of the Fokker-Planck Equation:

$$\frac{dH}{dt} = \frac{\partial^2 H}{\partial x^2}$$

Describes the spreading of particles due to random motion.

How would we simulate this?

Some Ideas

Every step move the particle left or right randomly. Keeping the variance over a given timescale constant, we decrease the step size to 0.

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Instead we could try some other probability distributions. Just need zero mean and constant variance over interval.

Central Limit Theorem tells us that these all give the same result!

Distribution of $x(t)$ will be a Gaussian with variance proportional to t.

Wiener Noise

Wiener process is differential *dW* with:

$$E(dW_t) = 0 \quad E(dW_t^2) = 1 \quad E(dW_t dW_s) = \delta(t-s)$$

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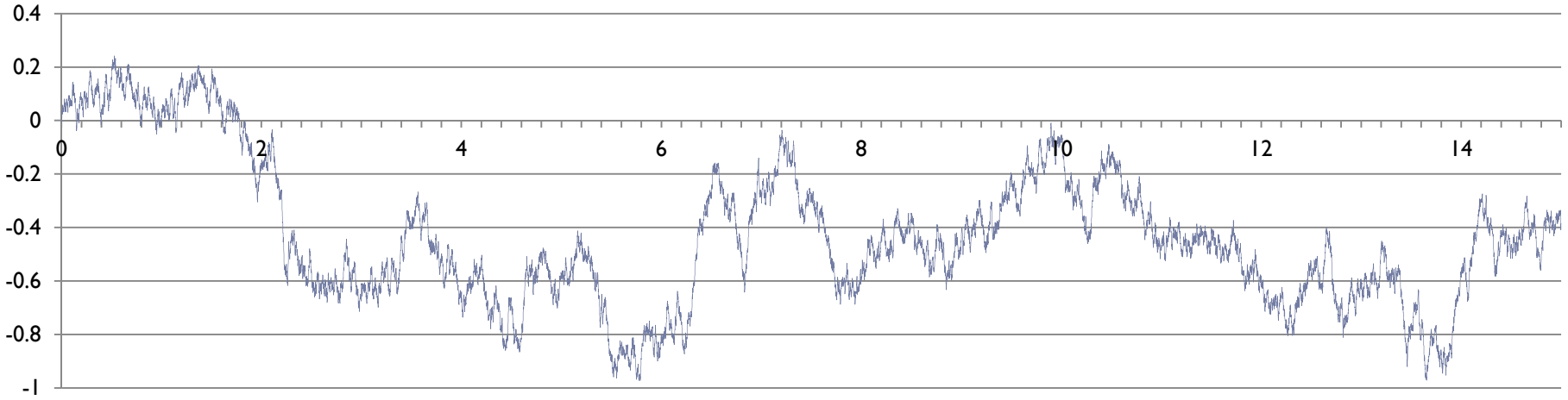
To solve diffusion equation, sample points and evolve via

$$dx = dW$$

To actually simulate this, vary *x* randomly with small intervals.

Wiener Noise

A typical Wiener process looks like



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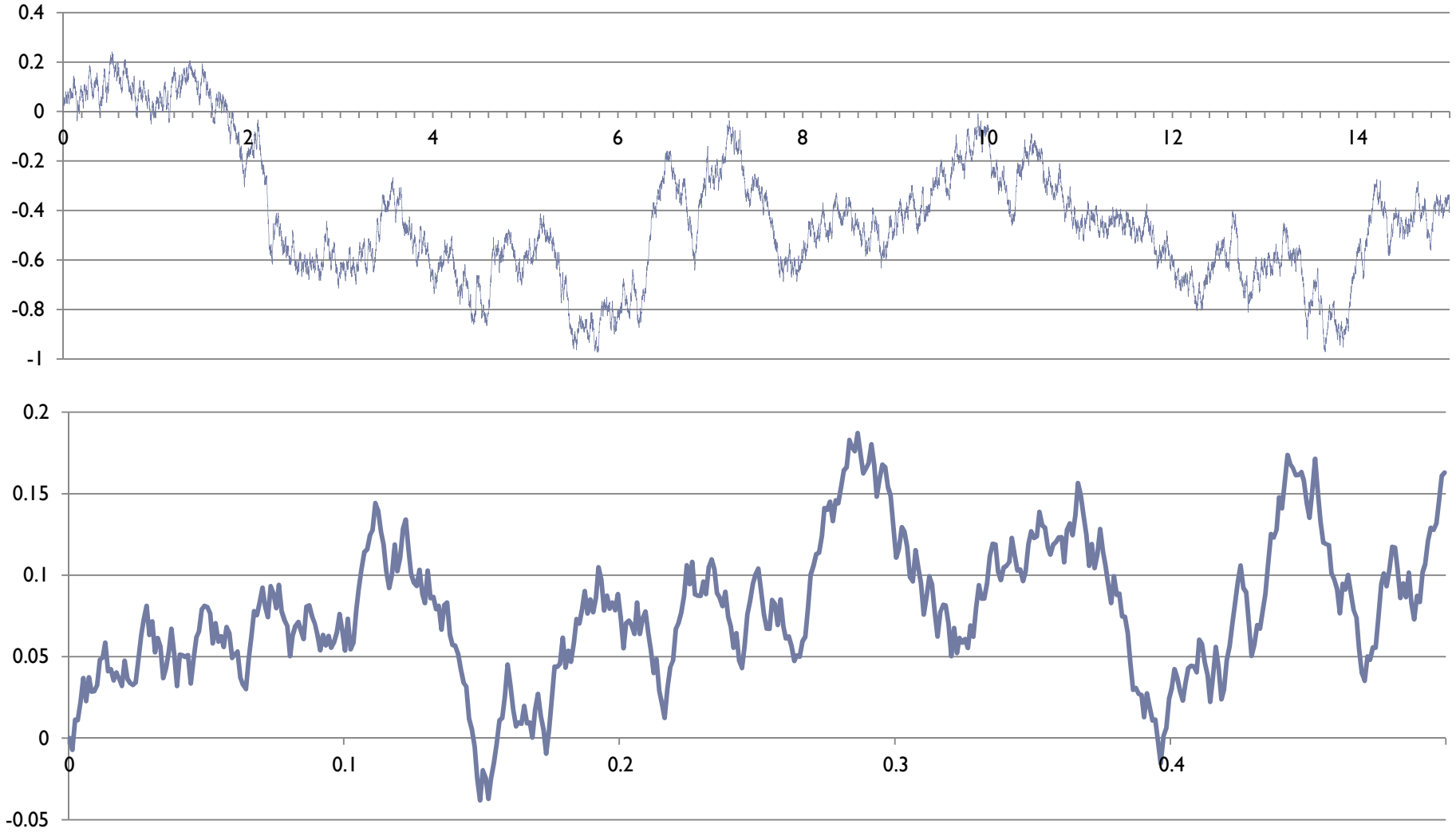

Fokker-Planck Unravellings

Since:

$$\frac{dH}{dt} = \frac{\partial^2 H}{\partial x^2} \quad \longrightarrow \quad dx = dW$$

we can generalise:

$$\frac{dW}{dt} = \frac{\partial}{\partial x_i \partial x_j} B_{ij} W \longrightarrow dx_i = C_{ij} dV_j$$

where $C_{ij}^T C_{ij} = B_{ij}$ and dV_j are Wiener processes.

Fokker-Planck Unravellings

This gives us the unravelling for the equation

$$\frac{dW}{dt} = \left(\frac{\partial}{\partial x_i} A_i + \frac{\partial}{\partial x_i \partial x_j} B_{ij} \right) W$$

as the equation:

$$dx_i = A_i dt + C_{ij} dV_j$$

Open classical systems follow the same equation.

Truncated Wigner Method

Most effective for large systems.

It is an uncontrolled approximation.

Distinctly quantum behaviour such as negative Winger functions do not occur.

Method won't work for Wigner functions with negative values.

Need B_{ij} to be positive-definite.

Wigner Functions in Quantum Optics

Quantum optics uses raising and lower operators:

$$\hat{a} = \hat{x} + i\hat{p} \qquad \hat{a}^T = \hat{x} - i\hat{p}$$

Phase-space becomes

$$\alpha = x + ip \qquad \alpha^* = x + ip$$

Operator correspondences are

$$\hat{a}\hat{\rho} \longrightarrow \left(\alpha + \frac{1}{2}\frac{\partial}{\partial \alpha^*}\right) W \qquad \hat{a}^T\hat{\rho} \longrightarrow \left(\alpha^* - \frac{1}{2}\frac{\partial}{\partial \alpha}\right) W$$

Wigner Functions in Higher Dimensions

Easy to extend to higher dimensions.

Grid based methods scale exponential

Truncated Wigner Method scales linearly.

Conclusion

The Wigner function provides a representation of quantum states and operators in phase space.

This provides one with a useful tool for visualising and studying quantum systems.

It provides a tool for probing the correspondence between classical and quantum systems.

Simulation methods such as the Truncated Wigner Method allow one to semiclassically simulate quantum systems stochastically and efficiently.